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Singular Workshop
Tutorial

1. Let $I = \langle y \cdot (x-1),\ z \cdot (x-1) \rangle \subset R = \mathbb{Q}[x,y,z]$.

   (a) Compute the dimension of $V(I)$, that is, compute $\dim(I)$, the Krull dimension of R/I.

   (b) Compute the dimension of $V(I)$ at $(0,0,0)$, that is, compute $\dim(I)$, the Krull dimension of T/IT, where $T = \mathbb{Q}[x,y,z]_{\langle x,y,z \rangle}$.

2. Sort all monomials of degree ≤ 2 in the variables x, y, z with respect to the monomial orderings lp, dp, and ls.

3. Let $I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[x,y,z]$ be the ideal generated by

$$f_1 = x^2 + y^2 - 1$$
$$f_2 = x^2 + z^2 - 1$$
$$f_3 = x + y + z$$

   (a) Compute the reduced Gröbner basis of I with respect to lp.

   (b) Deduce from the result, that $V(I)$ consists of 4 points.

   (c) Find one of the points.

4. Using Gröbner bases, show that for

$$C = \left\{ \left(t^2 - 1, t^3 - t\right) \mid t \in \mathbb{R} \right\} \subset \mathbb{A}^2_{\mathbb{R}}$$

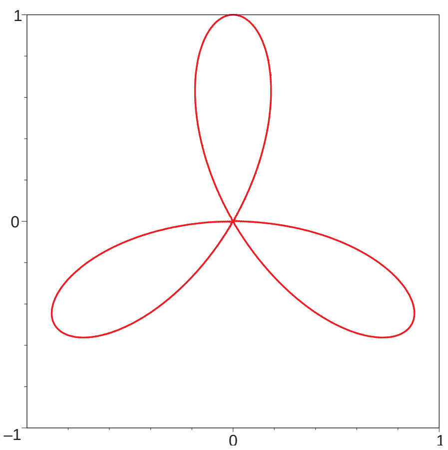
   we have

$$I(C) = \left\langle x^3 + x^2 - y^2 \right\rangle \subset \mathbb{R}[x,y].$$

   Make a plot of the curve C.

5. Compute and plot the closure of the image of the curve

$$C = V\left(y^3 - 3x^2y - (x^2+y^2)^2\right) \subset \mathbb{A}^2_{\mathbb{R}}$$



   under the rational map

$$\varphi = \left(\frac{\overline{x}^2}{\overline{y}^2}, \frac{\overline{x}}{\overline{y}} \right) : C \dashrightarrow \mathbb{A}^2_{\mathbb{R}}.$$

6. Let $A \in \mathbb{Q}^{n\times m}$ be a matrix. Using the SINGULAR command Gröbner basis command `std` for ideals, implement a procedure which computes the reduced row echelon form of A.

   You may also want to use the following commands, see the online help:

   `ring`, `matrix`, `proc`, `int`, `nrows`, `ncols`, `def`, `basering`, `imap`, `poly`, `ideal`, `for`, `option(redSB)`, `diff`, `maxideal`, `setring`, `return`.

7. Let $R = K[x_1, ..., x_n]$.

   (a) Using the SINGULAR command `eliminate`, write a procedure that computes $I \cap J$ for ideals $I, J \subset R$.

   (b) Suppose $J = \langle g \rangle \subset R$ is principal and $I \subset R$ is an ideal. Applying your ideal intersection command from (a), write a procedure which computes $I : J$.

   (c) For any two ideals $I, J \subset R$, write a procedure that computes $I : J$. Try out some examples.

8. Let

$$I = \left\langle xz - y^2,\ x - yz \right\rangle \subset \mathbb{R}[x, y, z]$$

   (a) Compute

$$J = I : \langle x \rangle$$

   and

$$I : J$$

   Hint: You can use your implementation from Exercise 7 or the SINGULAR command `quotient`.

   (b) Write I as an intersection of two prime ideals.

   (c) Find a polynomial parametrization of $V(J)$.

9. Consider the affine twisted cubic curve $C = V(I)$ with

$$I = \left\langle x_1^2 - x_2,\ x_1^3 - x_3 \right\rangle \subset K[x_1, x_2, x_3].$$

   (a) Compute the reduced Gröbner basis of $I^{\mathrm{h}} \subset R = K[x_0, x_1, x_2, x_3]$ with respect to *dp*.

   (b) Compute a free resolution of R/I^{h} as an R-module.

   (c) Determine the dimension, degree and arithmetic genus of the projective twisted cubic $\mathrm{pc}(C) = V(I^{\mathrm{h}}) \subset \mathbb{P}^3(K)$.

10. Let $R = K[x, y]$ and

$$f = \begin{pmatrix} xy + y^2 \\ xy^2 - 1 \\ x^2y + x^2 + xy^2 + xy \end{pmatrix}, \quad g_1 = \begin{pmatrix} y \\ 0 \\ xy + x \end{pmatrix}, \quad g_2 = \begin{pmatrix} x + 1 \\ y^2 \\ 0 \end{pmatrix} \in R^3$$

   Divide f by (g_1, g_2) using the monomial ordering extending the lexicographic ordering to R^3 by giving

   (a) priority to the monomials in R,

   (b) priority to the components.

11. In $R = K[x_1, ..., x_5]$, consider the ideal

$$I = \langle x_1x_3,\ x_2x_4,\ x_3x_5,\ x_4x_1,\ x_5x_2 \rangle,$$

and let $>$ be the ordering dp.

(a) Using the SINGULAR commands

```
resolution A = res(I,0);
resolution B = sres(I,0);
```

compute graded free resolutions of R/I.

(b) From the matrices `A[i]` and `B[i]` determine the respective Betti tables.

(c) From each of the Betti tables derive the Hilbert polynomial $P_{R/I}$. What do you observe?

(d) For the projective variety $X = V(I) \subset \mathbb{P}^4$, compute $\dim(X)$, $\deg(X)$, and $p_a(X)$.

12. Let $R = K[x_1, ..., x_n]$. Given ideals $I = \langle f_1, ..., f_a \rangle \subset R$ and $J = \langle g_1, ..., g_b \rangle \subset R$, consider the matrix

$$G = \begin{pmatrix} f_1 & \dots & f_a & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & g_1 & \dots & g_b & 1 \end{pmatrix} \in R^{2\times(a+b+1)},$$

and let $S = \operatorname{syz}(G) \in R^{(a+b+1)\times g}$ be a syzygy matrix of G.

(a) Show that the entries of the last row of S generate $I \cap J$.

(b) Applying this algorithm, compute the intersection

$$\langle x_1, x_2 \rangle \cap \langle x_1 - x_3, x_2 - x_3 \rangle \subset K[x_1, x_2, x_3].$$

Hint: You can use the SINGULAR command `syz`.

13. Let $R = K[x_0, .., x_2]$. For

$$A = \begin{pmatrix} x_1 & x_0 & 0 \\ 0 & x_2 & x_1 \end{pmatrix} \in R^{2\times 3}$$

and

$$B = \begin{pmatrix} x_1x_2 & x_0x_1 & 0 & 0 \\ 0 & 0 & x_1x_2 & x_0x_1 \end{pmatrix} \in R^{2\times 4}$$

compute a presentation of

$$\operatorname{subquot}(A, B) = \frac{\operatorname{im}(A) + \operatorname{im}(B)}{\operatorname{im}(B)}$$

that is, a matrix $C \in R^{3\times c}$ with

$$\operatorname{coker}(C) = \operatorname{subquot}(A, B).$$

14. Let $R = K[x_0, x_1, x_2]$ and

$$I = \left\langle x_0^3,\ x_1^3,\ x_0^2x_2^2,\ x_0x_1x_2^2,\ x_1^2x_2^2 \right\rangle \subset R$$

(a) Find the irredundant decomposition of I into irreducible ideals.

(b) Find an irredundant primary decomposition of I.

(c) Verify your result using SINGULAR.
Hint: You can use the command `primdecGTZ` from the library `primdec.lib`.

15. Write a SINGULAR function which computes the number of lattice points in $\Gamma^{-}(f) \cap \mathbb{R}^2_{\geq 0}$ for a bivariate polynomial $f \in \mathbb{C}[x,y]$.

16. Write a SINGULAR function which, if possible, computes the tropical variety of an ideal $I = \langle f, g \rangle \subseteq \mathbb{C}\{\{t\}\}[x,y]$ generated by a univariate polynomial $f \in C\{\{t\}\}[x]$ and a bivariate polynomial $g \in C\{\{t\}\}[x,y]$ using Newton polygons.