TROPICAL LINEAR SYSTEMS AND THE TROPICAL JACOBIAN

JANKO BÖHM

Abstract. These are the notes for a series of talks on tropical linear systems, and the tropical Abel-Jacobi theorem. They give examples and details on results from [1] and [2].

1. Tropical linear systems

1.1. **Introduction.** Given a divisor D on a compact Riemann surface C of genus $g = h^0\left(\Omega^1\right) = h^0(K)$, we ask to determine the dimension $h^0(D)$ of $H^0(C, \mathcal{O}_C(D))$, that is, the number of independent meromorphic functions f on C with

$$(f) + D \geq 0$$

The Riemann-Roch theorem tells us

Theorem 1.1.

$$h^0(D) = \deg(D) - g(C) + 1 + h^0(K - D)$$

Remark 1.2. The Riemann-Roch theorem gives a picture of the behaviour of the dimension of a generic linear system of an effective divisor

$$h^0(D) = \begin{cases} 1 & \text{for } \deg(D) \leq g(C) \\ \deg(D) - g(C) + 1 & \text{for } \deg(D) > g(C) \end{cases}$$

A divisor with $h^0(K - D) \neq 0$ is called special.

Remark 1.3. To illustrate the usefulness of the Riemann-Roch theorem, we recall how it implies some basic facts on curves:

If $g(C) \geq 2$ then the complete linear system $|K|$ has no base points: If $p \in C$ would be in the base locus of $|K|$, then

$$h^0(K - p) = h^0(K) = g(C)$$

Hence the Riemann-Roch theorem tells us, that

$$\begin{aligned} h^0(p) &= \deg(p) - g(C) + 1 + h^0(K - p) \\ &= 1 - g(C) + 1 + g(C) \\ &= 2 \end{aligned}$$

Hence there is a non-constant meromorphic function on C, which is holomorphic on $C - \{p\}$ and has a single pole at p. Hence S would be biholomorphic to $\mathbb{P}^1$, which has genus 0.

So K gives a morphism

$$\begin{array}{cccc} \iota_K: & C & \rightarrow & \mathbb{P}^{g(C)-1} \\ & p & \mapsto & (\omega_1(p) : ... : \omega_g(p)) \end{array}$$

where $\omega_1, ..., \omega_g$ are a basis of $H^0\left(C, \Omega^1\right)$.

This map is injective, if for all points $p, q \in C$ there is an $\omega \in H^0\left(C, \Omega^1\right)$ with

$$\omega(p) = 0,\ \omega(q) \neq 0$$

and it is an immersion, if for all $p \in C$ there is an $\omega \in H^0\left(C, \Omega^1\right)$ such that ω vanishes to order exactly 1 at p.

Hence ι_K is an embedding iff for all p, q

$$h^0(K - p - q) < \underbrace{h^0(K - p)}_{g(C)-1}$$

On the other hand, by the Riemann-Roch theorem, the left hand side is

$$h^0(K - p - q) = g(C) - 3 + h^0(p + q)$$

hence

$$h^0(K - p - q) < h^0(K - p) \Leftrightarrow h^0(p + q) = 1$$

Hence ι_K fails to be an embedding, iff there is a meromorphic function on C that has only two poles, that is, iff C is a two-sheeted covering of $\mathbb{P}^1$. Such a Riemann surface is called **hyperelliptic**.

1.2. **Tropical curves.**

Definition 1.4. For us, a **graph** Γ is a (finite) set $V(\Gamma)$ of vertices and a set $E(\Gamma)$ of edges which are unordered pairs of elements of $V(\Gamma)$, i.e., we allow edges connecting a vertex to itself.

The **valence** $\operatorname{val}(P)$ of a vertex P is the number of edges P is contained in.

A **metric graph**, is a graph together with a length function

$$l : E(\Gamma) \rightarrow \mathbb{R}_{>0}$$

Consider intervals $I_e = [0, l(e)] \subset \mathbb{R}$ for $e \in E(\Gamma)$ and glue I_{e_1} and I_{e_2} at end points, if $e_1 \cap e_2 \neq \emptyset$ give a topological space, called the **geometric realization** $|\Gamma|$ of Γ.

The first betti number of Γ is called the **genus** $g(\Gamma)$. It holds

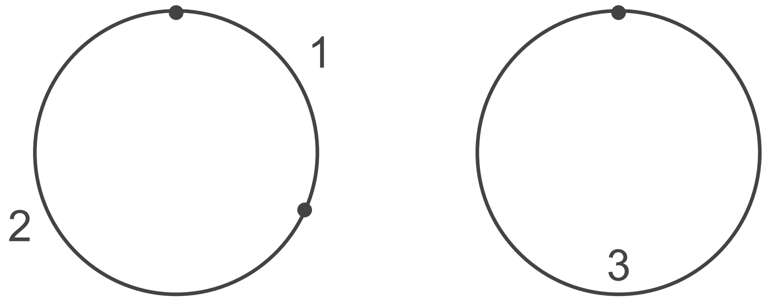
$$g(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + \#\text{connected components}$$

For us, a **tropical curve** is a connected metric graph Γ with $\operatorname{val}(P) \geq 2$ for all $P \in V(\Gamma)$.

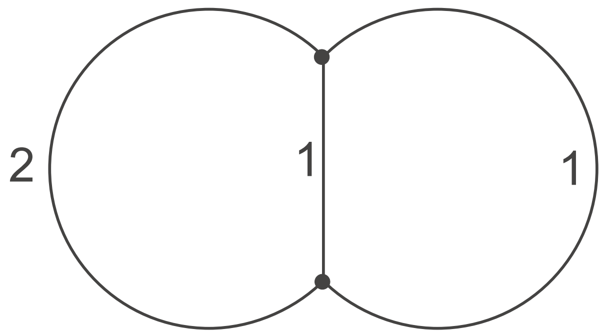
Two curves are called **equivalent**, if they represent the same metric space.

Example 1.5. Tropical curves of $g(C) = 0$

genus $g(C) = 1$



(which are equivalent) and $g(C) = 2$



Remark 1.6. We could also allow 1-valent vertices. Then we can consider

$$l : E(\Gamma) \to \mathbb{R}_{>0} \cup \{\infty\}$$

and have unbounded edges with a vertex at infinity and the edge is identified with $[0, \infty]$.

Remark 1.7. If these abstract tropical curves are embedded into a tropical toric variety

$$T(\mathrm{TV}(\Sigma)) = \frac{\mathrm{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)}, \mathbb{R}\right)}{\mathrm{Hom}_{\mathbb{R}}\left(A_{n-1}(\mathrm{TV}(\Sigma)) \otimes \mathbb{R}, \mathbb{R}\right)}$$

(of dimension n), we add (counted with multiplicity) degree many unbounded edges corresponding to rays $\Sigma(1)$.

1.3. **Divisors.**

Definition 1.8. A **divisor** on a tropical curve C is an element of the free abelian group $\mathrm{Div}\, C$ generated by the points of $|C|$, that is,

$$D = \textstyle\sum_i a_i P_i$$

with $a_i \in \mathbb{Z}$ and $P_i \in |C|$.

The **degree** of D is

$$\deg D = \textstyle\sum_i a_i$$

The divisor D is called **effective** if $a_i \geq 0$ for all i.

1.4. **Rational functions.**

Definition 1.9. A **rational function** on an open subset $U \subset |C|$ is a continuous piecewise linear function

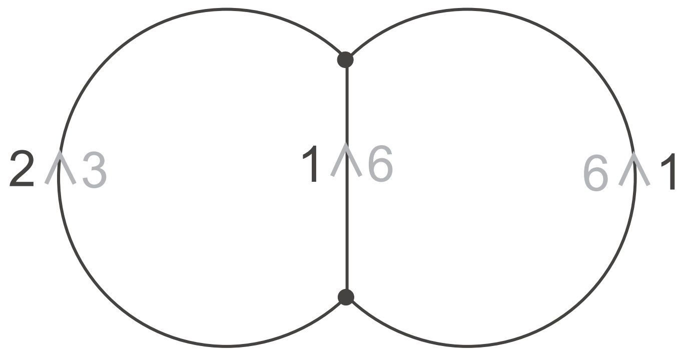
$$f : U \to \mathbb{R}$$

(with a finite number of pieces) with integer slopes.

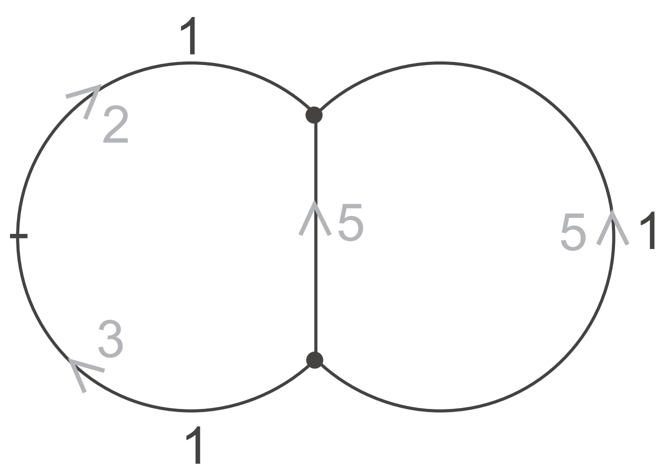
Denote by $\mathcal{M}(U)$ the set of rational functions on U.

If we allow for unbounded edges, then f may take values $\pm\infty$ at the unbounded edges.

Example 1.10. A rational function on $|C|$ is given (up to a constant) by specifying slopes:



The slope may also change in the interior of edges



1.5. **Principal divisors.**

Definition 1.11. Denote by t_i the coordinate on C given by an outward primitive tangent vectors at a point $P \in |C|$.

Given a rational function $f : U \to \mathbb{R}$ we define the **order** of f at P as

$$\mathrm{ord}_P(f) = \sum_i \frac{\partial f}{\partial t_i}(P)$$

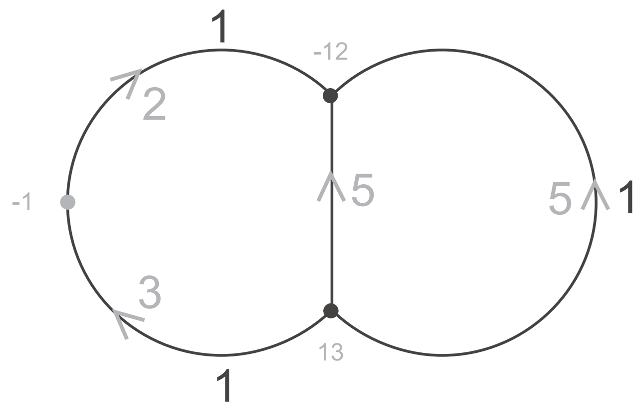
that is, the sum of all outgoing slopes.

A rational function $f : U \to \mathbb{R}$ is called **regular**, if $\mathrm{ord}_p(f) \geq 0$ for all $p \in U$.

Definition 1.12. Then the **principal divisor** of $f : |C| \to \mathbb{R}$ is

$$(f) = \sum_{P \in |C|} \mathrm{ord}_P(f)\, P$$

Example 1.13. The principal divisor of a rational function



Proposition 1.14. *The degree of a principal divisor of a rational function $f : |C| \to \mathbb{R}$ is*

$$\deg(f) = 0$$

Proof. As

$$\mathrm{ord}_P(f) = \sum_i \frac{\partial f}{\partial t_i}(P)$$

each slope appears in

$$\deg(f) = \sum_{P \in |C|} \mathrm{ord}_P(f)$$

twice (inward and outward) with opposite sign. $\square$

Corollary 1.15. *There is no non-constant regular function on $|C|$.*

1.6. **Canonical divisors.**

Definition 1.16. The **canonical divisor** of C is

$$K_C = \sum_{P \in V(C)} (\mathrm{val}(P) - 2)\, P$$

Note, that if a curve degenerates into $C_0 = \bigcup_i C_i$ then in the tropical curve

$$\sum_{j,\, j \neq i} C_i \cdot C_j = \mathrm{val}(C_i)$$

hence by $C_0.C_i = 0\ \forall i$ we have

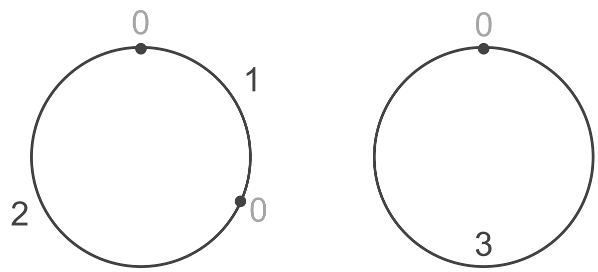
$$C_i.C_i = -\,\mathrm{val}(C_i)$$

By adjunction formula

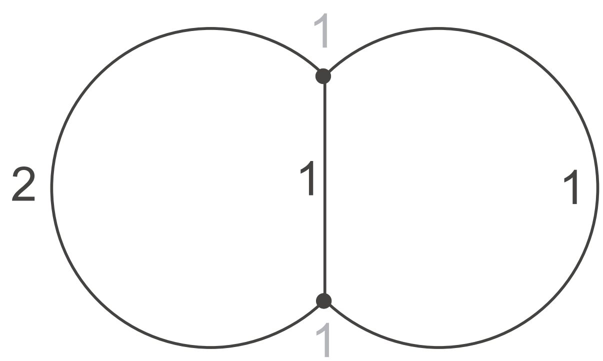
$$K_C.C_i = -C_i.C_i - 2 = \mathrm{val}(C_i) - 2$$

Example 1.17. Canonical divisors, genus $g(C) = 0$

genus $g(C) = 1$



and $g(C) = 2$



Remark 1.18. What is a meromorphic 1-form, the divisor of a meromorphic 1-form, and why is it a canonical divisor?

1.7. **Linear systems.**

Definition 1.19. For a divisor D on C define the space of global sections of D

$$\mathcal{L}(D) = \{f \in \mathcal{M}(C) \mid D + (f) \geq 0\}$$

and the corresponding **complete linear system**

$$|D| = \{D + (f) \mid f \in \mathcal{L}(D)\}$$

of divisors **linearly equivalent** to D (that is differing from D by a principal divisor).

The dimension of the linear system is defined as

$$\dim |D| = \max \{k \mid \mathcal{L}(D - P_1 - ... - P_k) \neq 0 \ \forall P_1, ..., P_k \in |C|\}$$

and $\dim |D| = -1$ if $\mathcal{L}(D) = 0$.

The space $\mathcal{L}(D)$ depends only on the metric space represented by C. Global rescaling of the metric structure of C and simultaneously of D does not change $\dim |D|$.

Remark 1.20. As $\deg(f) = 0$, all divisors in the linear system have the same degree

$$\deg(D + (f)) = \deg(D)$$

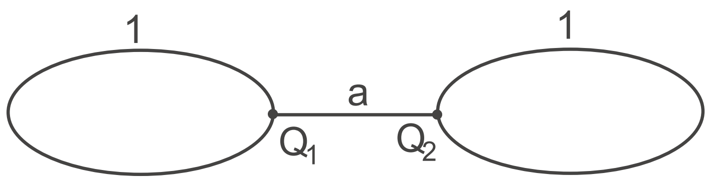
If $\deg(D) < 0$ then for all f we have $\deg(D + (f)) = \deg(D) < 0$ hence $D + (f) \ngeq 0$, so

$$\deg(D) < 0 \Rightarrow \dim|D| = -1$$

Otherwise

$$\dim|D| \leq \deg(D)$$

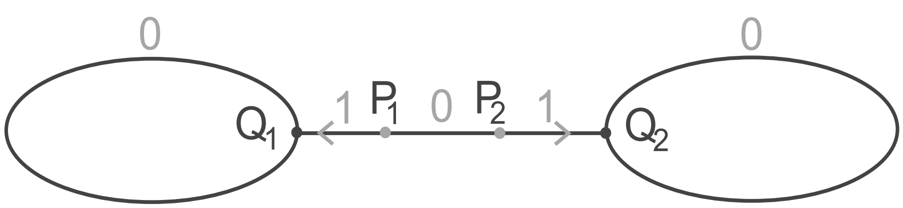
Example 1.21. Consider the canonical divisor $K_C = Q_1 + Q_2$ of the curve C



Suppose that

$$K_C + (f) = P_1 + P_2$$

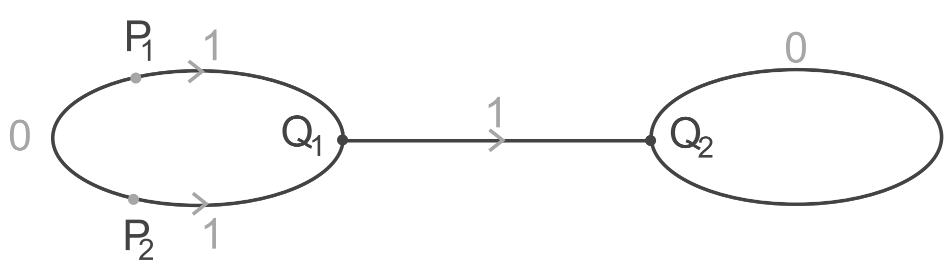
We can achieve any two points on the middle edge via the rational function with slopes



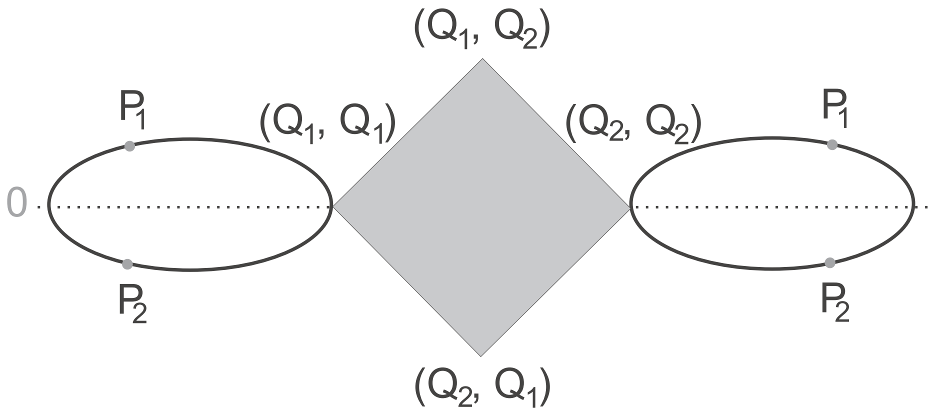
hence the set of all divisors linear equivalent to K_C is parametrized by

$$(P_1, P_2) \in [0, a]^2$$

We will show later that P_1 and P_2 cannot lie on the two different cycles. Suppose P_1 and P_2 lie on the cycle containing Q_1 then the continuity of the f implies that P_1 and P_2 have the same distance from Q_1. So we consider the rational function



As $P_1 + P_2 = P_2 + P_1$ we obtain $|D|$ as the S_2-quotient of



that is

Example 1.22. We now compute the dimension of the linear system in the previous Example 1.21:

For any $P_1 \in C$ there is an f with

$$(f) + K_C = P_1 + P_2$$

Then

$$(f) + K_C - P_1 = P_2 \geq 0$$

hence

$$f \in \mathcal{L}(K_C - P_1)$$

that is, $\dim |K_C| \geq 1$.

On the other hand there are P_1, P_2, for example

such that for all f

$$(f) + K_C \neq P_1 + P_2$$

As $(f) + K_C - P_1 - P_2 \neq 0$ but

$$\deg((f) + K_C - P_1 - P_2) = 0$$

we have

$$(f) + K_C - P_1 - P_2 \ngeq 0$$

So we conclude

$$\dim |K_C| = 1$$

Example 1.23. If we consider the following divisor $Q_1 + Q_2$ on the curve C from Example 1.21

we will see later that

$$(f) + Q_1 + Q_2 = P_1 + P_2$$

with P_i on the same loop as Q_i, hence, by continuity, f has to be constant, that is,

$$\mathcal{L}(Q_1 + Q_2) = \mathbb{R}$$

and

$$\dim |Q_1 + Q_2| = 0$$

Proposition 1.24. *Let D be a divisor on C. Then $\mathcal{L}(D)$ has the structure of a tropical semimodule, over the tropical semiring*

$$\mathbb{T} = (\mathbb{R}, \oplus, \odot)$$

with

$$a \oplus b = \max(a, b)$$
$$a \odot b = a + b$$

that is, $\mathcal{L}(D)$ is subset of

$$\mathbb{T}^{|C|} = \{|C| \to \mathbb{T}\}$$

which is closed under pointwise $\oplus$

$$\mathcal{L}(D) \times \mathcal{L}(D) \to \mathcal{L}(D)$$
$$f \oplus g = (P \mapsto f(P) \oplus g(P) = \max\{f(P), g(P)\})$$

and scalar multiplication

$$\mathbb{T} \times \mathcal{L}(D) \to \mathcal{L}(D)$$
$$\lambda \odot f = (P \mapsto \lambda \odot f(P))$$

Proof. Suppose $f, g \in \mathcal{L}(D)$ and $\lambda \in \mathbb{T}$. Then for the principal divisor of f it holds

$$(\lambda \odot f) = (\lambda + f) = (f)$$

that is

$$\lambda \odot f \in \mathcal{L}(D)$$

If $f(P) > g(P)$ then

$$\operatorname{ord}_P(f \oplus g) = \operatorname{ord}_P(f)$$

If $f(P) = g(P)$ then

$$\frac{\partial}{\partial t_i}(f \oplus g) = \max\left\{\frac{\partial f}{\partial t_i}, \frac{\partial g}{\partial t_i}\right\}$$

So in any case

$$\operatorname{ord}_P(f \oplus g) \geq \max\{\operatorname{ord}_P(f), \operatorname{ord}_P(g)\}$$

hence

$$\operatorname{ord}_P(f \oplus g) + D(P) > 0 \; \forall P \in |C|$$

(with the coefficient $D(P)$ of P in D), that is,

$$f \oplus g \in \mathcal{L}(D)$$

□

We already used the following lemma to calculate the dimension of the linear system in the above example:

Lemma 1.25. *Let D be a divisor of integer points on C, that is, of integer distance from the vertices, let*

$$D + (f) = P_1 + ... + P_n$$

(with P_i not necessarily distinct) and P_i a non-integer point of $|C|$ on a cycle C' of C. Then there is a second non-integer point $P_j \neq P_i$ with $P_j \in C'$.

Proof. We identify the cycle with the interval $[0, l(C')]$. Suppose there is only one non-integer point P_i on C'. If f has a multiple zero, the claim is obvious.

Now assume that f has a single zero. Let $x \in \mathbb{Z} \cap [0, l(C')]$ with

$$P_i \in [x - 1, x]$$

By adding a constant to f we can assume that

$$f(0), ..., f(x - 1) \in \mathbb{Z}$$

As f has integer slopes and $|P_i - x| \notin \mathbb{Z}$ and $\operatorname{ord}_P(f) = 1$

$$f(x), ..., f\left(l\left(C'\right)\right) \notin \mathbb{Z}$$

a contradiction to

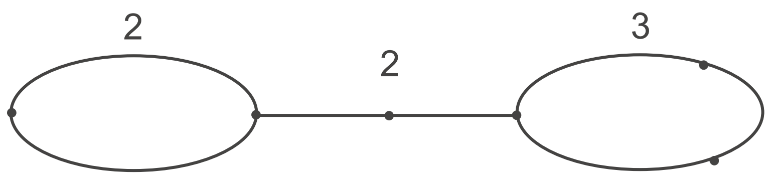
$$f(0) = f\left(l\left(C'\right)\right)$$

by continuity. □

1.8. **Integer curves.** Suppose C is a tropical curve with

$$l : E(C) \to \mathbb{Z}_{>1}$$

Introducing 2-valent vertices, we can assume that all edges have length one, for example



Remark 1.26. If

$$(f) = \sum_i a_i P_i$$

is a divisor of integer points of C then the function f is determined by linear interpolation of the values of f at the vertices. The principal divisor of f can be described by the formula

$$(f) = \sum_{(P,Q)\in E(C)} (f(P) - f(Q))(P - Q)$$

(note that the terms are independent of the ordering of the tuple (P, Q)).

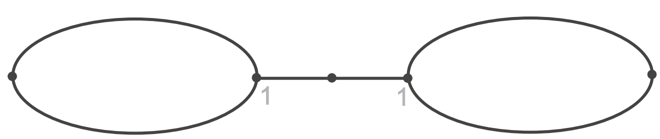
We can use this formula to compute the discrete linear system

$$\tilde{\mathcal{L}}(D) = \{f \in \mathcal{M}(C) \mid D + (f) \geq 0,\ D + (f) \text{ integer divisor}\}$$
$$\widetilde{|D|} = \left\{D + (f) \mid f \in \tilde{\mathcal{L}}(D)\right\}$$

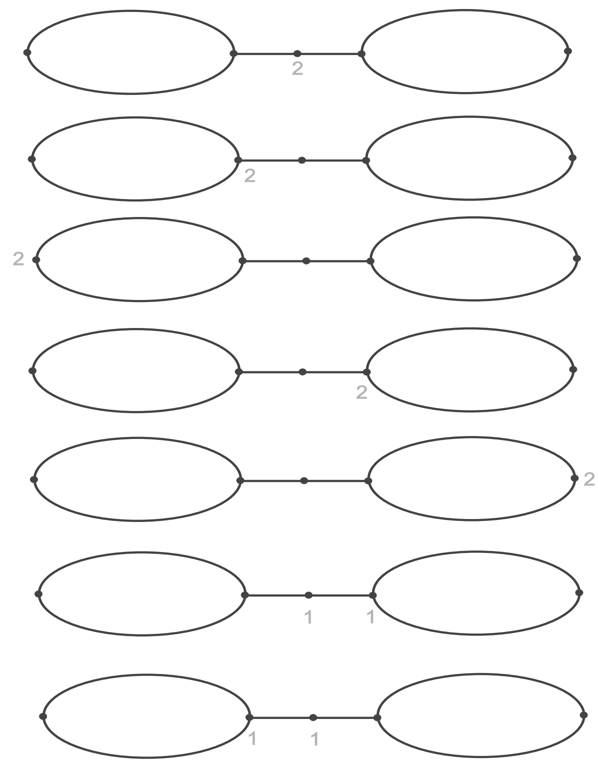
and its dimension

$$\tilde{r}(D) = \max\left\{k \mid \begin{array}{l} \forall P_1, ..., P_k \in V(C)\, \exists f : V(C) \to \mathbb{Z} \text{ with} \\ D + (f) - P_1 - ... - P_k \geq 0 \end{array}\right\}$$

Example 1.27. Starting with



we can achieve all the following configurations



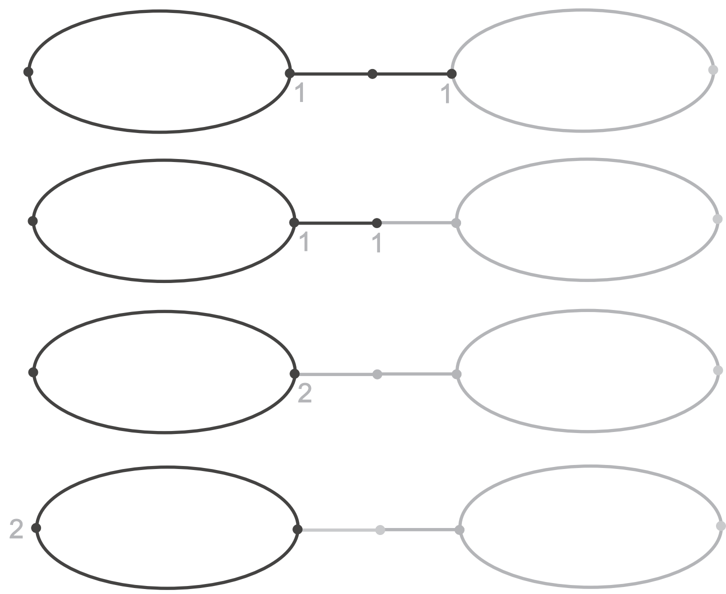
using appropriate integer functions (as constructed already above). These configurations form $\widetilde{|K_C|}$.

Using this we observe that for all $P \in V(C)$ there is an $f : V(C) \to \mathbb{Z}$ with $(f) + D - P \geq 0$. On the other hand there are configurations of two points, which cannot be achieved. We conclude again

$$\tilde{r}(K_C) = 1$$

The formula for (f) represents **chip-firing**: Given a subgraph with a divisor of degree d_i on a boundary point P_i of external valency r_i, we can simultaneously move one point along each edge emanating from the subgraph at P_i provided $d_i \geq r_i \ \forall i$.

Example 1.28. We use chip-firing to relate the linear equivalent divisors of the previous Example 1.27 (shading the subgraph):



1.9. **Riemann-Roch theorem.** We now prove the theorem of Riemann-Roch using the corresponding result of Baker and Norine in the case of non-metric graphs.

Theorem 1.29. *For an integer divisor D on an integer tropical curve C, we have*

$$\tilde{r}(D) - \tilde{r}(K_C - D) = \deg(D) + 1 - g(C)$$

We want to show from this:

Theorem 1.30. *For a divisor D on a tropical curve C, we have*

$$\dim|D| - \dim|K_C - D| = \deg(D) + 1 - g(C)$$

First of all, we approximate by a curve C with

$$l : E(C) \to \mathbb{Q}_{>0}$$

and a rational divisor. Rescaling the curve we may assume

$$l : E(C) \to \mathbb{Z}_{>1}$$

and D integer. Rescaling further we get

$$\tilde{r}(D) = \dim|D|$$
$$\tilde{r}(K_C - D) = \dim|K_C - D|$$

by Lemma 1.32, which follows from the following Lemma 1.31 :

Lemma 1.31. *Let D be integer on integer C. If there is an f with $(f)+D \geq 0$, then there is an f such that*

$$(f)+D \geq 0$$

and $(f)+D$ is integer.

Proof. We prove the claim by induction on the degree $m = \deg D$. For $m < 0$ nothing is to show. Write

$$(f)+D = P_1 + ... + P_m$$

If $m = 0$ then $(f)+D = 0$ is integer.

If $m > 0$ then

$$(f)+D-P_1 = P_2 + ... + P_m \geq 0$$

hence

$$\mathcal{L}(D-P_i) \neq 0$$

If some P_i is integer, then by the induction hypothesis

$$\tilde{\mathcal{L}}(D-P_i) \neq 0$$

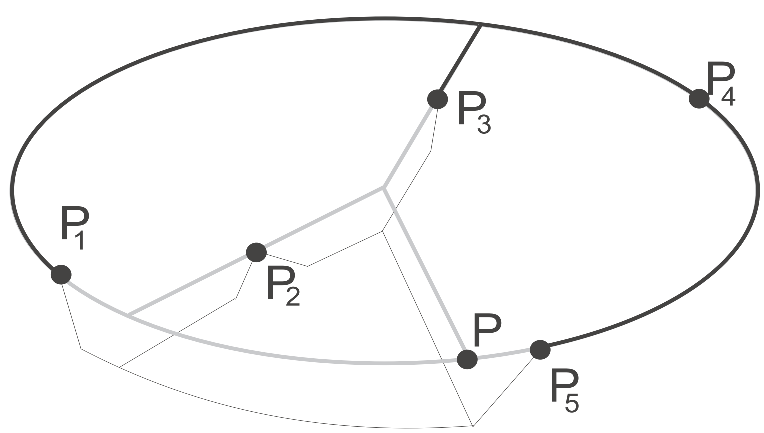
and hence $\tilde{\mathcal{L}}(D) \neq 0$.

Suppose all P_i are not integer: We may assume that P_m has among all P_i the minimal distance from an integer vertex P of C. Consider the function

$$h : |C| \to \mathbb{R}$$

$$Q \mapsto \begin{cases} -\min\{\|P_m - P\|, \|Q-P_i\| \mid i\} & \text{if } Q \text{ is in the conn. comp. of } P \\ 0 & \text{otherwise} \end{cases}$$

Example:



Then

$$\begin{aligned} f+h \in \mathcal{L}(D-P) \\ &\Leftrightarrow \underbrace{(f+h)}_{(f)+(h)} + D - P \geq 0 \\ &\Leftrightarrow (h) + P_1 + ... + P_m - P \geq 0 \end{aligned}$$

Assume this divisor has a summand which is a negative multiple of Q.
First of all, $Q \neq P$ as $\mathrm{ord}_P(h) \geq 1$.
Hence h has a pole at Q, so $Q = P_i$ for some i, so $\mathrm{ord}_Q(h) = -2$.
This is only possible if Q is in the interior of the connected component of P.
So the connected component contains a cycle and Q is the only point of $(f) + D$ on the cycle.
This gives a contradiction to Lemma 1.25.
By the induction hypothesis

$$\tilde{\mathcal{L}}(D - P) \neq 0$$

and hence $\tilde{\mathcal{L}}(D) \neq 0$. □

Lemma 1.32. *Let D be integer on an integer C. Then there is an $N \geq 1$ such that on any multiple of $N \cdot C$ it holds*

$$\dim |D| = \tilde{r}(D)$$

Proof. Let $m = \dim |D| + 1$. For all $P_1, ..., P_{m-1}$

$$\mathcal{L}(D - P_1 - ... - P_{m-1}) \neq 0$$

hence by the previous Lemma 1.31

$$\tilde{\mathcal{L}}(D - P_1 - ... - P_{m-1}) \neq 0$$

so by definition

$$\tilde{r}(D) \geq m - 1 = \dim |D|$$

For the other inequality:
If $\dim |D| + 1 > \deg(D)$ (that is, $\dim |D| = \deg(D)$) then

$$\tilde{r}(D) \leq \deg(D) \leq \dim |D|$$

If $m = \dim |D| + 1 \leq \deg(D) = n$ consider the map

$$\pi_m : \{(f, P_1, ..., P_n) \mid D + (f) = P_1 + ... + P_n\} \to C^n \to C^m$$
$$(f, P_1, ..., P_n) \mapsto (P_1, ..., P_n) \mapsto (P_1, ..., P_m)$$

As $\mathrm{image}(\pi_m) \subset C^m$ is closed, and strictly smaller (as $m > \dim |D|$), there is a

$$(P_1, ..., P_m) \notin \mathrm{image}(\pi_m)$$

with rational coordinates. Rescale by

$$N = \mathrm{lcm}(\mathrm{denom}(\mathrm{dist}(P_i, C \cap \mathbb{Z})) \mid i)$$

By construction

$$\mathcal{L}(D - P_1 - ... - P_m) = 0$$

hence also

$$\tilde{\mathcal{L}}(D - P_1 - ... - P_m) = 0$$

so

$$\tilde{r}(D) \leq m - 1 = \dim |D|$$

□

Example 1.33. For $D = K_C$ we obtain

$$\begin{aligned}\dim|K_C| &= \deg(K_C) + 1 - g(C)\\ &= \sum_{P\in V(C)} (\mathrm{val}_P(C) - 2) + 1 - g(C)\\ &= 2|E(C)| - 2|V(C)| + 1 - g(C)\\ &= 2g(C) - 2 + 1 - g(C)\\ &= g(C) - 1\end{aligned}$$

as

$$g(C) = |E(C)| - |V(C)| + 1$$

and

$$\sum_{P\in V(C)} \mathrm{val}_P(C) = 2|E(C)|$$

So we recover in Example 1.21

$$\dim|K_C| = 2 + 1 - 2 = 1$$

2. The tropical Jacobian

2.1. Tropical abelian varieties.

Definition 2.1. Consider $\mathbb{R}^g$ with the lattice $\mathbb{Z}^g$. A **tropical torus** is a quotient $\mathbb{R}^g/\Lambda$ by a lattice $\Lambda \subset \mathbb{R}^g$.

A **polarized tropical abelian variety** is a tropical torus together with a homomorphism

$$\Lambda \to (\mathbb{Z}^g)^*$$

such that the corresponding bilinear map

$$\mathbb{R}^g \times \mathbb{R}^g \to \mathbb{R}$$

is positive definite symmetric.

It is called **principally** polarized if $\Lambda \to (\mathbb{Z}^g)^*$ is an isomorphism.

2.2. Holomorphic 1-Forms.

Definition 2.2. The tangent space T_pC of C at p is the set of derivations $\frac{\partial}{\partial t_i}$ of C at p corresponding to the tangent directions t_i of C at p. A **holomorphic differential form** on C is a collection of maps

$$\omega_p : T_pC \to \mathbb{R}$$

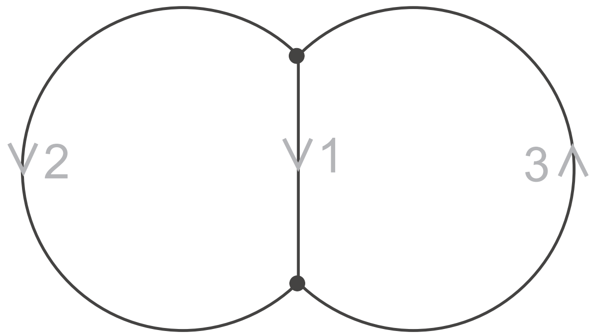
such that

$$\textstyle\sum_i \omega_p\left(\frac{\partial}{\partial t_i}\right) = 0$$

for all p.

Denote by $\Omega^1(C)$ the set of all holomorphic 1-forms, and by $\Omega^1_{\mathbb{Z}}(C)$ the space of forms taking integer values.

Example 2.3. A global holomorphic 1-form on a curve of genus 2 specifying the value of the form on tangent vectors.

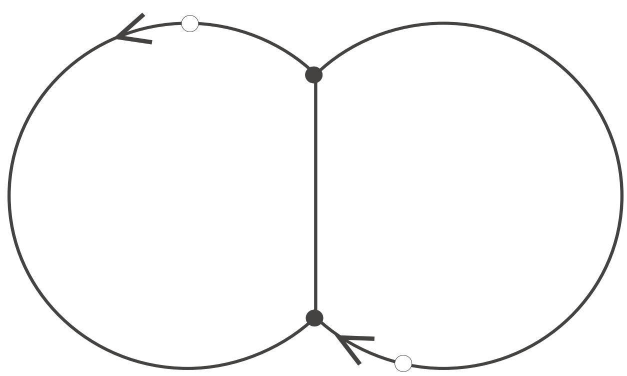


Note that this is not a rational function. It is a flow on the curve, the value of the form on the tangent vector is the volume of the flow per unit of time.

Definition 2.4. Let C be a curve of genus g. A set of **break points** of C is a set of points $P_1, ..., P_g \in |C|$ of valency 2 together with a choice of an outward primitive integer tangent vector $\frac{\partial}{\partial t_i}$ at t_i for all i, such that $|C| \setminus \{P_1, ..., P_g\}$ represents a connected tree.

A choice of break points is equivalent to the choice of a connected fundamental domain $T \subset |C|$.

Example 2.5. A choice of break points for the curve in Example 2.3



A choice of break points specifies an isomorphism of $\mathbb{R}$-vector spaces

$$\begin{array}{cccc} \Phi: & \Omega^1(C) & \rightarrow & \mathbb{R}^g \\ & \omega & \mapsto & \left(\omega_{P_1}\left(\frac{\partial}{\partial x_1}\right), ..., \omega_{P_g}\left(\frac{\partial}{\partial x_g}\right)\right) \end{array}$$

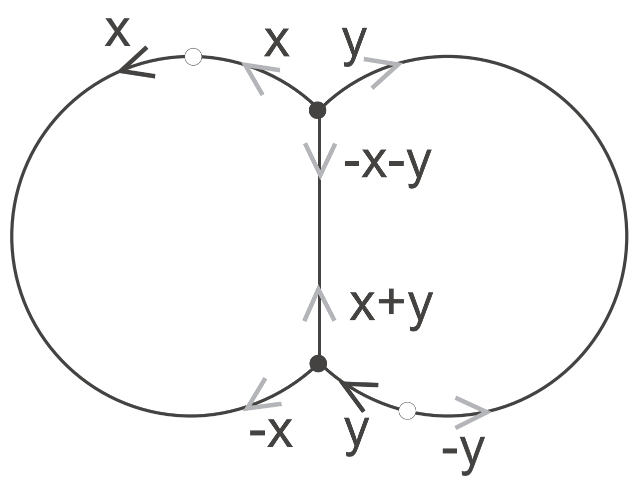
and a basis of the lattice

$$\Omega^1_{\mathbb{Z}}(C) \subset \Omega^1(C)$$

(as in the definition of $\Phi(\omega)$ we consider values on primitive integer tangent vectors), that is, the standard basis of

$$\mathbb{Z}^g \subset \mathbb{R}^g$$

Example 2.6. For the choice of break points in Example 2.3 the bijection is given by associating to the form



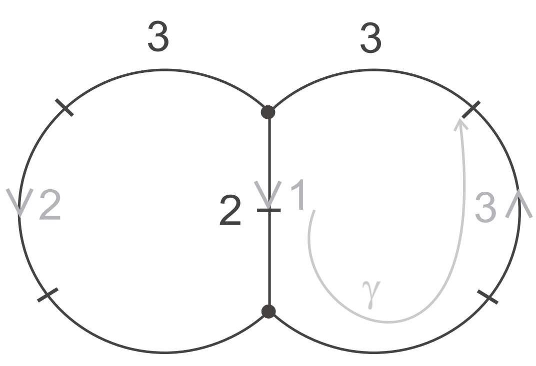
the tuple $(x, y) \in \mathbb{R}^2$.

2.3. **The tropical Jacobian.** For any path γ in C and $\omega \in \Omega^1(C)$ we can define the integral

$$\int_\gamma \omega \in \mathbb{R}$$

by pulling back the tropical 1-form to a classical 1-form on the interval.

Example 2.7. We compute the integral of the form given in Example 2.3 (also specifying the metric structure on the curve) over the path γ



as

$$\int_\gamma \omega = 1 \cdot 1 + 2 \cdot 3 = 7$$

Let

$$\Omega^1(C)^* = \mathrm{Hom}_{\mathbb{R}}\left(\Omega^1(C), \mathbb{R}\right) \supset \Omega^1_{\mathbb{Z}}(C)^* \cong (\mathbb{Z}^g)^*$$

be the space of $\mathbb{R}$-valued linear functionals on $\Omega^1(C)$. We obtain a $\mathbb{Z}$-monomorphism from the cycles to $\Omega^1(C)^*$

$$H_1(C,\mathbb{Z}) \hookrightarrow \Omega^1_{\mathbb{Z}}(C)^* \subset \Omega^1(C)^*$$
$$\gamma \mapsto \int_\gamma = \left(\omega \mapsto \int_\gamma \omega\right)$$

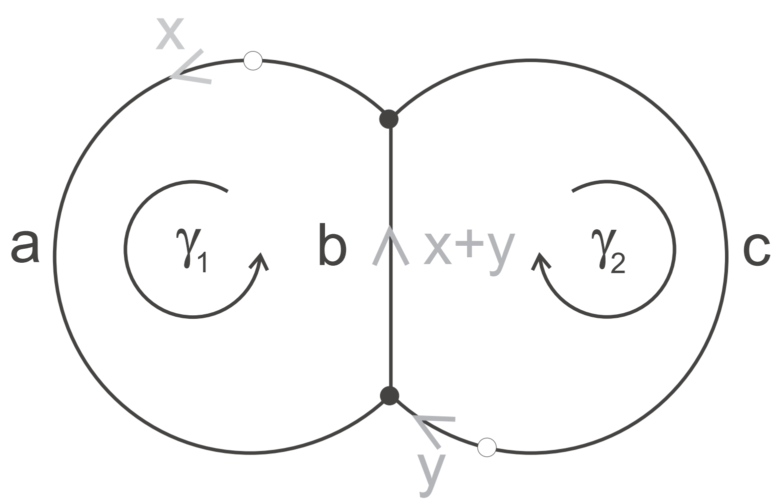
Definition 2.8. The Jacobian of the tropical curve C is

$$J(C) = \Omega^1(C)^* / H_1(C,\mathbb{Z})$$

By a choice of break points $H_1(C,\mathbb{Z})$ corresponds to a lattice $\Lambda \subset \mathbb{R}^g \cong \Omega^1(C)^*$ of rank g, and

$$J(C) \cong \mathbb{R}^g/\Lambda$$

Example 2.9. Consider the curve (metric structure with lengths a, b, c)



with the depicted choice of break points (specifying $\mathbb{Z}^g \subset \mathbb{R}^g$). Integrating over the cycle γ_1 we get

$$\int_{\gamma_1} \Phi^{-1}(x,y) = b(x+y) + ax = (a+b,b)\cdot(x,y)$$

that is

$$\int_{\gamma_1} = (a+b,b)\cdot$$

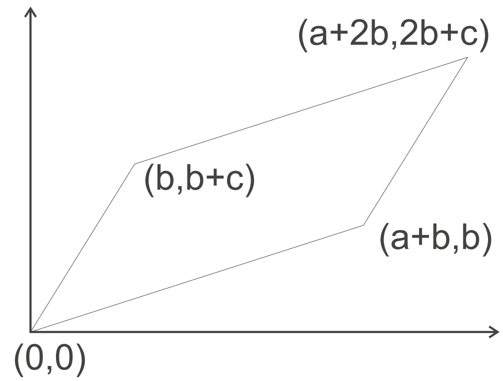
For γ_2 similarly

$$\int_{\gamma_2} \Phi^{-1}(x,y) = b(x+y) + cy = (b,b+c)\cdot(x,y)$$

that is

$$\int_{\gamma_2} = (b,b+c)\cdot$$

Hence the lattice Λ is generated by these two points



and $J(C)$ is the quotient.

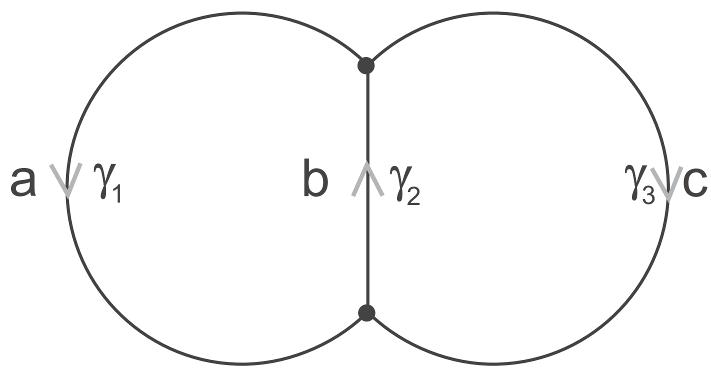
Consider the bilinear map

$$Q : \text{Paths}(C) \times \text{Paths}(C) \to \mathbb{R}$$

defined by extending bilinearly the definition for any non-selfintersecting paths γ

$$Q(\gamma, \gamma) = \text{length}(\gamma)$$

Example 2.10. Consider the paths $\gamma_1 + 2\gamma_2 + \gamma_3$ and $\gamma_1 + \gamma_2$ on



Then

$$Q(\gamma_1 + 2\gamma_2 + \gamma_3, \gamma_1 + \gamma_2) = a + 2b$$

The map Q induces a symmetric bilinear form

$$Q : H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \to \mathbb{R}$$

(as any zero-homologous cycle is trivial).

Lemma 2.11. *The induced bilinear map*

$$\Omega^1(C)^* \times \Omega^1(C)^* \to \mathbb{R}$$

is positive definite.

Proof. For

$$\gamma = \sum\nolimits_E a_E E$$

we have

$$Q(\gamma,\gamma) = \sum\nolimits_E a_E^2 \operatorname{length}(E)$$

□

From Q we obtain a map

$$\begin{array}{lllll} \tilde{Q}: & \Omega^1(C)^* & \to & \Omega^1(C)^{**} & \cong & \Omega^1(C) \\ & \int_\gamma & \mapsto & \left(\int_{\gamma'} \mapsto Q(\gamma,\gamma')\right) & & \\ & & & \left(\int_{\gamma'} \mapsto \int_{\gamma'}\omega\right) & \mapsto & \omega \end{array}$$

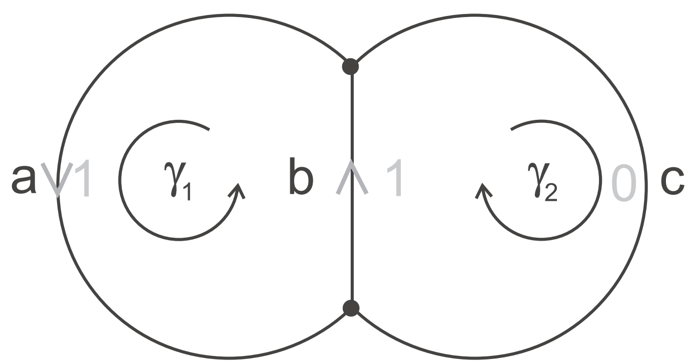
which restricts to

$$\begin{array}{cccc} \tilde{Q}: & \Omega^1_{\mathbb{Z}}(C)^* & \to & \Omega^1_{\mathbb{Z}}(C) \\ & \cong & & \cong \\ & \Lambda & & \mathbb{Z}^g \end{array}$$

Example 2.12. We compute the image $\tilde{Q}(\gamma_1) \in \Omega^1_{\mathbb{Z}}(C)$, that is, we find an $\omega \in \Omega^1_{\mathbb{Z}}(C)$ such that

$$(Q(\gamma_1,\gamma_1), Q(\gamma_1,\gamma_2)) \cdot \;= (a+b,b) \cdot \;= \left(\int_{\gamma_1}\omega, \int_{\gamma_2}\omega\right) \cdot$$

The form ω is given by taking flow 1 along the cycle γ_1 and 0 otherwise:



In the same way, for a basis of $\Lambda \cong H_1(C,\mathbb{Z})$ such that any cycle in the basis contains exactly one break point, the basis is mapped to the standard basis of $\mathbb{Z}^g$, hence

$$\tilde{Q}: \Omega^1_{\mathbb{Z}}(C)^* \to \Omega^1_{\mathbb{Z}}(C)$$

is an isomorphism, so:

Proposition 2.13. *$J(C)$ is a principally polarized tropical abelian variety.*

2.4. **Tropical Abel-Jacobi theorem.** Denote by $\mathrm{Div}^d(C)$ the set of degree d divisors and by $\mathrm{Pic}^d(C)$ its quotient by linear equivalence. Fix $P_0 \in C$. For

$$D = \sum_i a_i D_i \in \mathrm{Div}^d(C)$$

define $\tilde{\mu}(D) \in \Omega^1(C)^*$ by

$$\begin{array}{rccc} \tilde{\mu}(D): & \Omega^1(C) & \to & \mathbb{R} \\ & \omega & \mapsto & \sum_i a_i \int_{p_0}^{p_i} \omega \end{array}$$

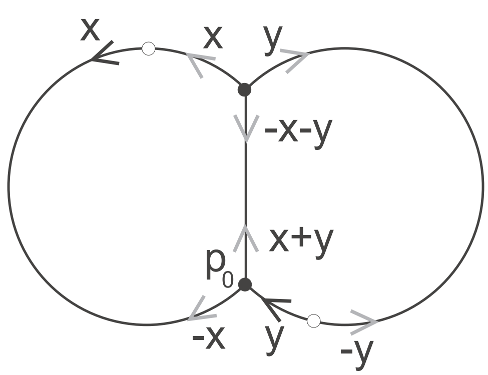
by fixing paths from p_0 to p_i.

For a path in $H_1(C,\mathbb{Z})$ we have $\tilde{\mu}(D) \in \Lambda$, hence we obtain a well defined map

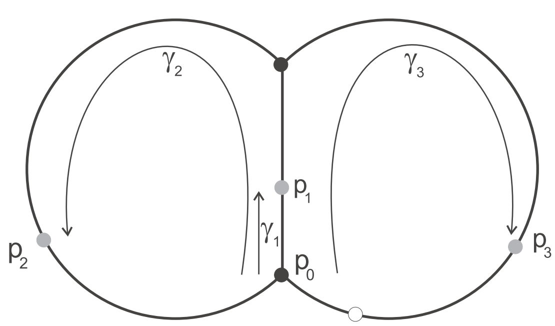
$$\begin{array}{rccc} \mu: & \mathrm{Div}^d(C) & \longrightarrow & J(C) \cong \mathbb{R}^g/\Lambda \\ & D & \mapsto & \tilde{\mu}(D) \end{array}$$

Remark: If $d > 0$ choice of P_0 corresponds to the Jacobi inversion constant $\varkappa$ in Theorem 2.19.

Example 2.14. We compute the image $\mu(C)$ of all points of C in the case of Example 2.9. Consider the choice of break points and P_0 as follows:



Recall that $(x,y) \in \mathbb{R}^2$ corresponds to a 1-form $\Phi^{-1}(x,y)$. Consider paths γ_i of length l_1, $b+l_2$ and $b+l_3$ as follows



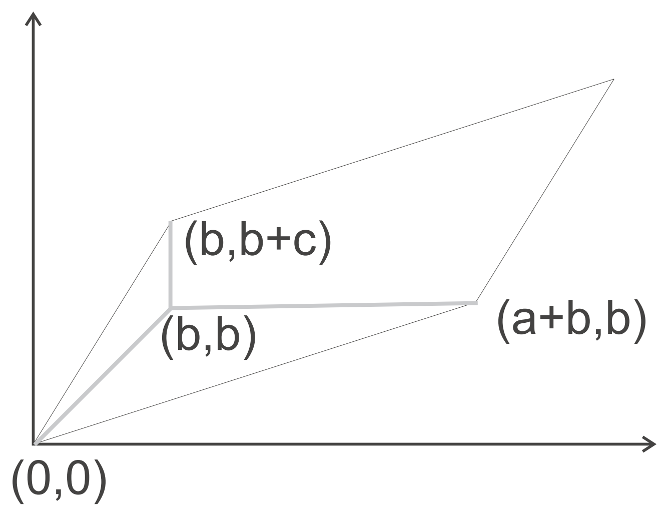
We obtain the integrals

$$\int_{\gamma_1} \Phi^{-1}(x,y) = l_1(x+y) = (l_1,l_1)\cdot(x,y)$$
$$\int_{\gamma_2} \Phi^{-1}(x,y) = b(x+y) + l_2 x = (b+l_2,b)\cdot(x,y)$$
$$\int_{\gamma_3} \Phi^{-1}(x,y) = b(x+y) + l_3 y = (b,b+l_3)\cdot(x,y)$$

hence

$$\mu(P_1) = (l_1,l_1)\cdot$$
$$\mu(P_2) = (b+l_2,b)\cdot$$
$$\mu(P_3) = (b,b+l_3)\cdot$$

and we obtain $\mu(C)$ as



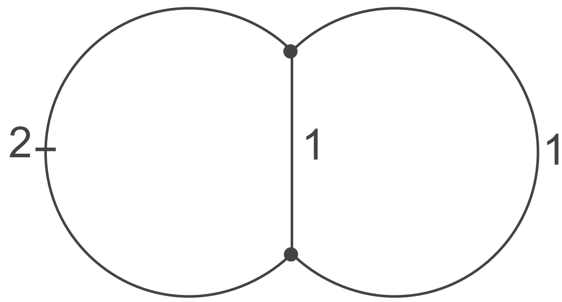
Theorem 2.15 (Tropical Abel theorem)**.** *For any d the map μ factors through* $\mathrm{Pic}^d(C)$

$$\begin{array}{ccc} \mathrm{Div}^d(C) & \longrightarrow & \mathrm{Pic}^d(C) \\ & \mu \searrow & \downarrow \phi \\ & & J(C) \end{array}$$

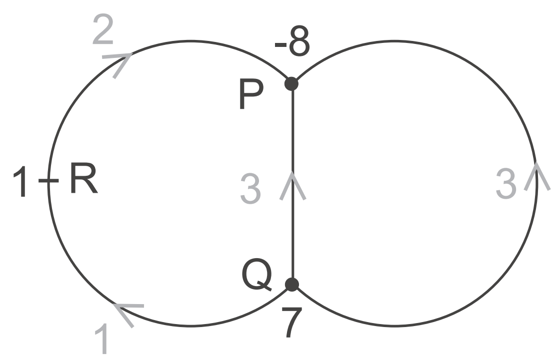
and ϕ is injective.

Example 2.16. We illustrate $\mu(D) = 0$ for $D = (f)$ with $f \in \mathcal{M}(C)$ at an example. The general case works the same way.

Consider the curve



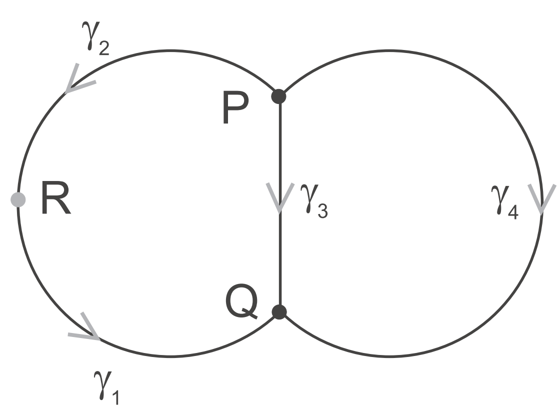
and the rational function f specified by



with divisor

$$D = (f) = 7Q - 8P + R = 7(Q - P) + (R - P)$$

In terms of the following paths



and choosing the base point $P_0 = P$ we have

$$\mu(D) = 7\int_{\gamma_3} + \int_{\gamma_2} + \Lambda$$

Using the cycles

$$\begin{aligned} \gamma_1 + \gamma_2 - \gamma_3 &= 0 \\ \gamma_3 - \gamma_4 &= 0 \end{aligned}$$

we get

$$\begin{aligned} \mu(D) &= 7\int_{\gamma_3} + \int_{\gamma_2} + \Lambda \\ &= 4\int_{\gamma_3} + 3\int_{\gamma_4} + \int_{\gamma_2} + \Lambda \\ &= \int_{\gamma_1} + 2\int_{\gamma_2} + 3\int_{\gamma_3} + 3\int_{\gamma_4} + \Lambda \\ &=: \int_{\gamma} + \Lambda \\ &= Q\left(\tilde{Q}^{-1}(-), \gamma\right) = 0 \end{aligned}$$

where

$$\gamma = \gamma_1 + 2\gamma_2 + 3\gamma_3 + 3\gamma_4$$

is the path associated to f.

We now give the general proof of the statement in Theorem 2.15 that μ factors through $\mathrm{Pic}^d(C)$.

Proof. Let $f \in \mathcal{M}(C)$ and

$$(f) = D = \sum_i P_i - \sum_i Q_i$$

Define

$$\mathrm{path}(f) = \sum_i a_i \gamma_i$$

where γ_i is a path of constant slope a_i of f.

We claim

$$\mu(D) = \sum_i \int_{Q_i}^{P_i} + \Lambda = \int_{\mathrm{path}(f)} + \Lambda$$

Choose break points B_i (different from the P_j, Q_j) and a basis $\delta_1, ..., \delta_g$ of $H_1(C, \mathbb{Z})$ such that any δ_i contains exactly one B_i.

Denote by ε_i the unique paths from Q_i to P_i avoiding all B_j. We have to show that

$$\sum_i \varepsilon_i = \mathrm{path}(f) \bmod H_1(C, \mathbb{Z})$$

Consider the curve C' with $C' = C \setminus \{B_1, ..., B_g\}$. We show that

$$\sum_i \varepsilon_i = \mathrm{path}\left(f'\right) = \mathrm{path}(f) + \sum_j \frac{\partial f}{\partial t_j}(B_j)\,\delta_j$$

for

$$f' = f + \sum_j \frac{\partial f}{\partial t_j}(B_j) \cdot t_j \in \mathcal{M}(C')$$

The second equality is clear by construction of f'.

For the first equality note that $(f') = (f) \subset |C'|$ as divisors on C' and C, respectively. Furthermore f' is non-zero only on the ε_i.

Then

$$\mu(D) = \int_{\text{path}(f)} + \Lambda = Q\left(\tilde{Q}^{-1}(-), \text{path}(f)\right) = 0$$

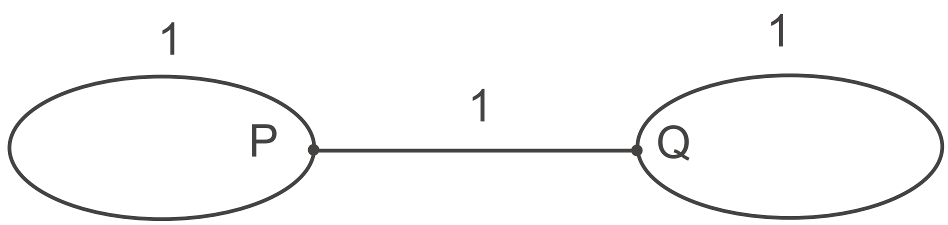
The last expression is zero as in path (f) each path γ_i occurs with multiplicity equal to the slope of f, hence paired with a cycle we sum up the differences of function values of f along, hence get 0. □

To put it differently, we are integrating the exact form df over a cycle.

Example 2.17. For the other direction we consider a divisor D of $\deg D = 0$ and with $\mu(D) = 0$ and construct an $f \in \mathcal{M}(C)$ with

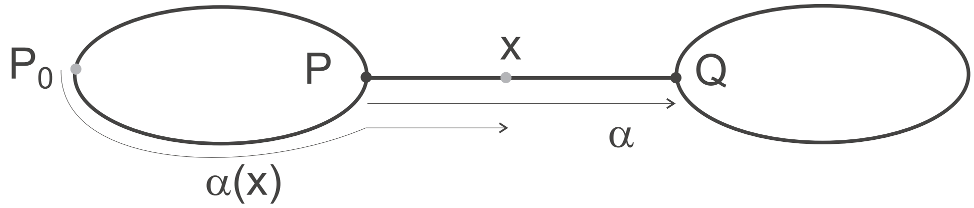
$$(f) = D$$

Take, for example, $D = P - Q$ on



As any 1-form is zero on the middle edge, we have $\int_P^Q = 0$ hence $\mu(D) = 0$.

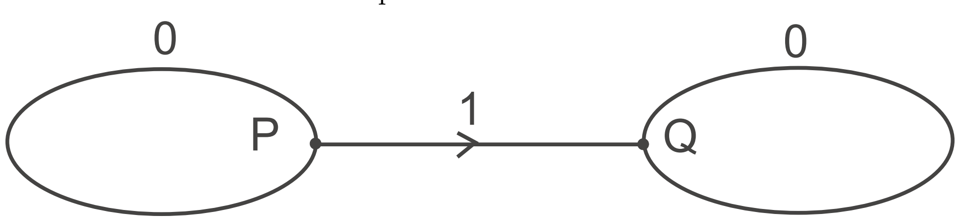
Consider the base point P_0 and paths α from P to Q and $\alpha(x)$ from P_0 to x as follows



Then

$$f(x) = Q(\alpha, \alpha(x))$$

is a rational function with slopes



and $(f) = P - Q$.

The function f is independent of the choice of $\alpha(x)$ as for any cycle δ

$$Q(\alpha,\delta) = \int_\alpha \tilde{Q}(\delta) = \mu(D)\left(\tilde{Q}(\delta)\right) = 0$$

by assumption (note that only for cycles $\tilde{Q}(\delta)$ is defined).

Proof. Now assume

$$D = \textstyle\sum_i P_i - \sum_i Q_i \in \mathrm{Div}^0(C)$$

and $\mu(D) = 0$. We construct an $f \in \mathcal{M}(C)$ with

$$D = (f)$$

Choose paths α_i from P_i to Q_i such that

$$\sum_i \int_{\alpha_i} = 0$$

Then take

$$f(x) = \sum_i Q(\alpha_i, \alpha(x))$$

for a choice of a path $\alpha(x)$ from P_0 to x. Again f is independent of the choice of $\alpha(x)$ as for any cycle δ we have

$$\sum_i Q(\alpha_i,\delta) = \sum_i \int_{\alpha_i} \tilde{Q}(\delta) = 0$$

□

2.5. **Theta functions.** The tropical Laurent series

$$\Theta(x) = \max_{\lambda\in\Lambda}\left\{Q(\lambda,x) - \frac{1}{2}Q(\lambda,\lambda)\right\}$$

in $x \in \mathbb{R}^g$ has a Λ-periodic corner locus as the value of Θ changes under translation in Λ by a affine linear function in x:

Lemma 2.18. *For any $\mu \in \Lambda$ we have*

$$\Theta(x+\mu) = \Theta(x) + Q(\mu,x) - \frac{1}{2}Q(\mu,\mu)$$

Proof. Inside the maximum we have

$$\begin{aligned}
&Q(\lambda, x+\mu) - \frac{1}{2}Q(\lambda,\lambda) \\
&= Q(\lambda-\mu,x) + Q(\mu,x) + Q(\lambda,\mu) - \frac{1}{2}Q(\lambda-\mu,\lambda-\mu) - Q(\mu,x) - \frac{1}{2}Q(\mu,\lambda) \\
&= Q(\lambda-\mu,x) - \frac{1}{2}Q(\lambda-\mu,\lambda-\mu) + Q(\mu,x) + \frac{1}{2}Q(\mu,\mu)
\end{aligned}$$

which implies

$$\begin{aligned}\Theta(x+\mu) &= \max_{\lambda\in\Lambda}\left\{Q(\lambda,x+\mu)-\frac{1}{2}Q(\lambda,\lambda)\right\}\\ &= \max_{\lambda\in\Lambda}\left\{Q(\lambda-\mu,x)-\frac{1}{2}Q(\lambda-\mu,\lambda-\mu)\right\}\\ &+ Q(\mu,x)+\frac{1}{2}Q(\mu,\mu)\\ &= \max_{\lambda'\in\Lambda}\left\{Q\left(\lambda',x\right)-\frac{1}{2}Q\left(\lambda',\lambda'\right)\right\}\\ &+ Q(\mu,x)+\frac{1}{2}Q(\mu,\mu)\end{aligned}$$

□

So we can associate to Θ a Λ-periodic tropical hypersurface $\operatorname{trop}\Theta\subset\mathbb{R}^g$ and hence a well defined tropical hypersurface, i.e., divisor

$$\operatorname{trop}\Theta\subset\mathbb{R}^g/\Lambda$$

2.6. **Jacobi inversion.** For $\lambda\in\mathbb{R}^g$ denote by $\Theta_\lambda(x):=\Theta(x-\lambda)$ the translated theta function and $\operatorname{trop}\Theta_\lambda$ its divisor in $J(C)$. Let $D_\lambda=\mu^*\operatorname{trop}\Theta_\lambda$ the pull back of $\operatorname{trop}\Theta_\lambda$ to C via the Abel-Jacobi map $\mu:C\to J(C)$. Without proof we state:

Theorem 2.19. *For any $\lambda\in J(C)$ the divisor D_λ is effective of degree g.*

There is a universal $\varkappa\in J(C)$ such that

$$\mu(D_\lambda)+\varkappa=\lambda \text{ for all } \lambda\in J(C)$$

Hence ϕ is bijective.

References

[1] A. Gathmann, and M. Kerber, *A Riemann-Roch theorem in tropical geometry*, arXiv:math/0612129v2 [math.CO].

[2] G. Mikhalkin, and Ilia Zharkov, *Tropical curves, their Jacobians, and theta functions*, arXiv:math/0612267v2 [math.AG].

Department of Mathematics, Universität des Saarlandes, Campus E2 4, D-66123 Saarbrücken, Germany

E-mail address: boehm@math.uni-sb.de