Constructing Calabi-Yau mirrors via tropical geometry

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October 2009

Calabi-Yau varieties and mirror symmetry

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Supersymmetric string theory (unify gravity + QM)

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$$X$$

$\swarrow$		$\searrow$
B-model of X	$\times$	A-model of X
B-model of $X^\vee$		A-model of $X^\vee$

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Deformations of

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Tangent spaces

$H^1(T_X) = H^{2,1}(X)$	$\cong$	$H^{1,1}(X^\vee)$
by Bogomolov-Tian-Todorov if		Moser

$H^0(T_X) = H^{2,0}(X) = H^{1,0}(X) = 0$

Calabi-Yau varieties and mirror symmetry

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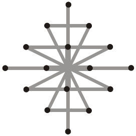

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
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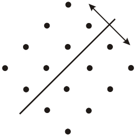

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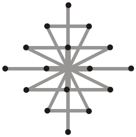
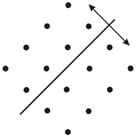


$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$



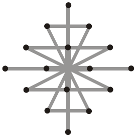
$$H^{2,1}(X) \quad \cong \quad H^{1,1}(X^\vee)$$

induces equality of Yukawa couplings

$$\langle -,-,- \rangle \quad = \quad \langle -,-,- \rangle$$

(from Picard-Fuchs differential equations)

(from # of $g = 0$ curves in given homology class)



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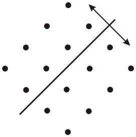

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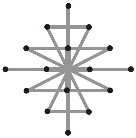
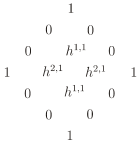
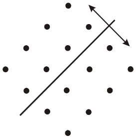
$\searrow$ $\swarrow$

Tropical geometry







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interpreting lattice points as

Deformations

Divisor classes
$(H^1(X^\vee, \mathcal{O}^*_{X^\vee}) = H^2(X^\vee, \mathbb{Z}))$

Degenerations

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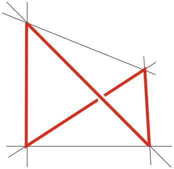

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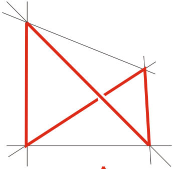

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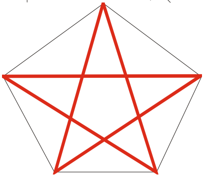

$$X_0 = \{x_0x_1 = x_1x_2 = x_2x_3 = x_3x_4 = x_4x_0 = 0\} \subset \mathbb{P}^4$$

X_t by structure theorem of Buchsbaum-Eisenbud

General setup

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Flat family of Calabi-Yau varieties $\mathfrak{X} \longrightarrow \operatorname{Spec} \mathbb{C}[t]$ with fibers $X_t \subset Y$.
Y a $\mathbb{Q}$-Gorenstein toric Fano variety defined by $\Sigma = \operatorname{Fan}(\Delta^*)$ in $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $N = \mathbb{Z}^n$.

$$\begin{array}{l} \operatorname{Strata}(X_0) \cong \mathit{Sphere} \\ \quad \cap \\ \operatorname{Strata}(Y) = \Delta \end{array}$$

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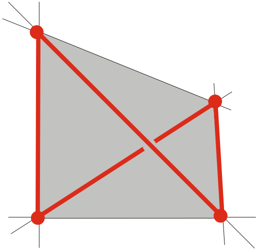

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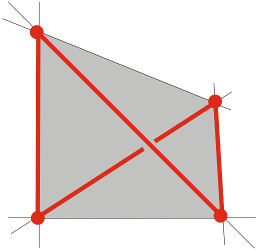

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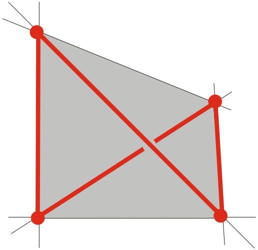

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Example

(Loading quartic.gif)

Example

(Loading torus.gif)

Tropical geometry and degenerations

Space of homogeneous weight vectors on S:

Tropical geometry and degenerations

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$$N_{\mathbb{R}} = \frac{\mathrm{Hom}\left(\mathbb{R}^{\Sigma(1)},\mathbb{R}\right)}{\mathrm{Hom}(A_{n-1}(Y)\otimes\mathbb{R},\mathbb{R})} .$$

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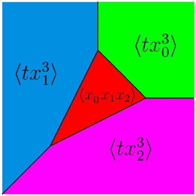

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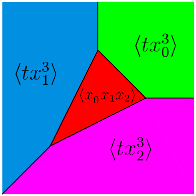

Domains of linearity of $\mathrm{trop}\,(f)$ $\cap \{w_t = 1\}$

Denote the tropical variety of $I \subset \mathbb{C}\,[t] \otimes S$

$$BF\,(I) = \mathrm{val}\left(V_{\mathbb{C}\{\{s\}\}}\,(I)\right) \subset \mathbb{R} \oplus N_{\mathbb{R}}$$

as the Bergman fan of I (considering t as a variable).

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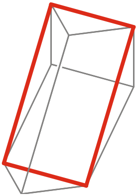

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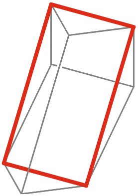

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$T_{I_0}(I)$ is subcomplex of $\partial\nabla$ of same dim and codim as X_t.

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Decompose the generators of I into characters m of $(\mathbb{C}^*)^{\Sigma(1)}$.

Example:

$$\begin{aligned} f_1 &= x_0 x_3 + t \cdot \left(x_0^2 + x_0 x_1 + \ \dots\right) \\ f_2 &= x_1 x_2 + t \cdot \left(x_0^2 + x_0 x_1 + \ \dots\right) \end{aligned} \quad \longleftrightarrow \quad \begin{matrix} \frac{x_0}{x_3}, \frac{x_1}{x_3}, \dots \\ \frac{x_0^2}{x_1 x_2}, \frac{x_0}{x_2}, \dots \end{matrix}$$

Represent 1st-order deformations $\varphi_m : I_0 \to S / I_0$

Homogeneous $\Longrightarrow$ $m \in \operatorname{image}\left(0 \to M \to \mathbb{Z}^{\Sigma(1)}\right)$

If base smooth: $\nabla = \operatorname{convhull}(\text{preimages})$

$$T_{I_0}(I) = \text{faces } F \text{ of } \nabla \text{ s.t.}$$

$$\left\langle m_0 + t \cdot \sum_{m \in F^*} a_m \varphi_m(m_0) \mid m_0 \in I_0 \right\rangle \subset \mathbb{C}[t]/\langle t^2 \rangle \otimes S$$

contains no monomial

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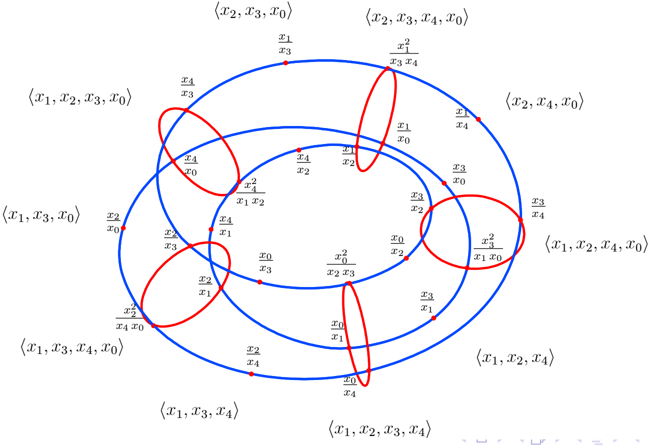

Topology of the special fiber tropical variety via limit map

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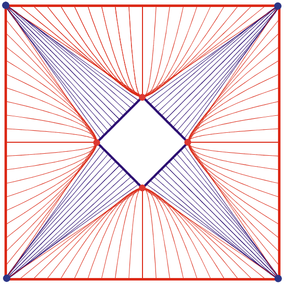

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Birational Fano $\hat{Y} = X(\hat{\Sigma}) \to Y$ such that

$$\begin{array}{rlcl} \forall F \in T_{I_0}(I) & \exists \sigma \in \hat{\Sigma} & : & F \subset \sigma \\ \dim(\sigma) = & \dim(F) + \operatorname{codim} X_t & & \end{array}$$

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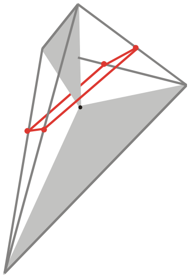

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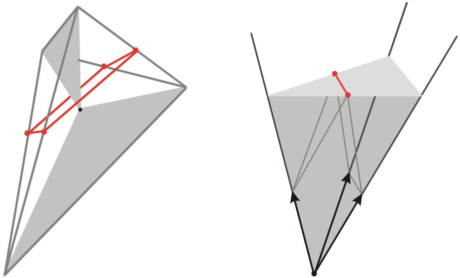

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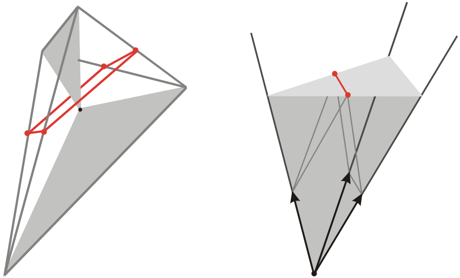

Hypersurface in $\mathbb{Q}$-Gor Y: $\quad \hat{\Sigma} = \mathrm{Fan}(\Delta^*_{-K_Y})$.

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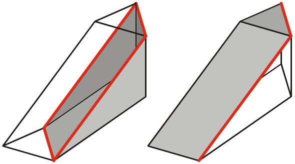

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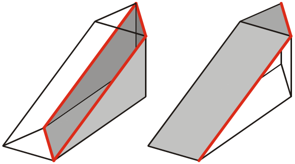

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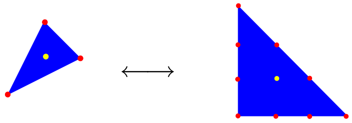
Recover the construction of Batyrev for hypersurfaces in Gorenstein toric Fano varieties (reflexive polytopes)

$$\nabla = \Delta^* \qquad \qquad \Delta = \nabla^*$$



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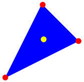

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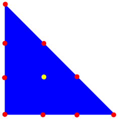
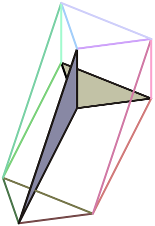


Recover the construction of Batyrev and Borisov for complete intersections in Gorenstein toric Fano varieties (nef partitions).

$$\nabla = \nabla_1 + \nabla_2 \qquad \Delta = \Delta_1 + \Delta_2$$



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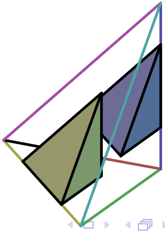

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- Pfaffian non-complete intersection mirrors

$$\begin{pmatrix} 0 & ty_4^2 & y_1y_2 & -y_5y_6 & ty_3^2 \\ -ty_4^2 & 0 & t(y_5-y_6) & y_3 & -y_7 \\ -y_1y_2 & -t(y_5-y_6) & 0 & -ty_7 & y_4 \\ y_5y_6 & -y_3 & ty_7 & 0 & t(y_1+y_2) \\ -ty_3^2 & y_7 & -y_4 & -t(y_1+y_2) & 0 \end{pmatrix}$$

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