Motivic Integration Seminar
Stringy Hodge Numbers and Motivic Integration

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Abstract

These notes are the manuscript for two talks given in a seminar on motivic integration. They try to explain the basics of motivic integration, how to apply this to the problem of defining Batyrev´s stringy E-function for varieties with canonical singularities and how the stringy E-function relates to the Hodge numbers of crepant resolutions. As a corollary to the transformation rule of the motivic integral we also prove, that birational Calabi-Yau have equal Hodge numbers. Be aware, that these are rough notes written to prepare the talks. Suggestions, comments and corrections are very much appreciated (boehm@math.uni-sb.de).

Contents

1 Introduction

Definition 1 *A normal projective d-dimensional algebraic variety X is called a **Calabi Yau variety**, if*

- *it has at worst Gorenstein canonical singularities.*
- $K_X = \mathcal{O}_X$
- $h^i(X, \mathcal{O}_X) = 0$ *for* $0 < i < d$.

Remark 2 *The Hodge diamond of a smooth Calabi-Yau 3 fold has the form*

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1}(X) & & 0 & \\
1 & & h^{1,2}(X) & & h^{1,2}(X) & & 1 \\
 & 0 & & h^{1,1}(X) & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

by Serre duality, Hodge duality and $K_X = \Omega_X^3 = \mathcal{O}_X$

$$H^{0,i}(X) \simeq H^i(X, \mathcal{O}_X) \simeq H^i\left(X, \Omega_X^3\right) \simeq H_{\bar{\partial}}^{3,i}(X)$$

So the Euler number is

$$\chi(X) = 2\left(h^{1,1}(X) - h^{2,1}(X)\right)$$

There is a perfect pairing given by wedge product (non canonical)

$$\Omega_X^1 \times \Omega_X^2 \to \Omega_X^3 \cong \mathcal{O}_X$$

so

$$T_X \cong \Omega_X^2$$

hence $H^0(X, T_X) \cong H^0(X, \Omega_X^2)$ *and* $H^1(X, T_X) \cong H^1(X, \Omega_X^2) = H_{\bar{\partial}}^{2,1}(X)$.

Definition 3 *Two smooth Calabi-Yau d folds X and X^* are called a* ***topological mirror pair****, if their Hodge numbers satisfy*

$$h^{p,q}(X) = h^{d-p,q}(X^*) \ \ \forall 0 \le p, q \le d \tag{1}$$

Remark 4 *If X and X^* form a topological mirror pair, then the Hodge diamonds is mirror symmetric with respect to the diagonal.*

Mirror symmetry interchanges the Hodge duality and Poincare duality.

Remark 5 *Some background from Physics:*
There are 5 different types of string theories. From the point of view of Physics mirror symmetry of two Calabi-Yau 3-folds X and X^* is the duality of two of these types of string theories, defined on the product of a Minkowski space and X resp. X^*.

Example for duality in physics in the simplest case: Maxwell´s equations describing the electromagnetic interaction are invariant under the transformation

$$E \mapsto Bc^2$$
$$B \mapsto -E$$

which shows, that the electrostatic theory for high interaction energies is equivalent to magnetic theory for low interaction energies. In the case of mirror symmetry the duality allows the treatment of enumerative problems in algebraic geometry.

In Physics the Euler number of X has an interpretation as the number of generations of Fermions in the corresponding string theory:

$$\textit{Number of generations} = \frac{1}{2}|\chi(X)| = \left|h^{1,1}(X) - h^{2,1}(X)\right|$$

In particular we see, that mirror symmetric string theories have the same number of Fermion generations.

Example 6 *Consider the family of quintic* 3 *folds* $X \subset \mathbb{P}^4$. *By adjunction formula*

$$K_X = \mathcal{O}_X$$

By Lefschetz Hyperplane Theorem we have

$$H^k(X, \mathbb{C}) \xleftarrow{\simeq} H^k\left(\mathbb{P}^4, \mathbb{C}\right) = \left\{ \begin{array}{ll} \mathbb{C} & k = 0, 2 \\ 0 & k = 1 \end{array} \right\}$$

hence as X *is Kähler we have* $h^{0,1}(X) = h^{0,2}(X) = 0$ *and*

$$h^{1,1}(X) = 1$$

From the Euler sequence and conormal sequence we get

$$h^{1,2}(X) = 101$$

Since $h^{1,1}(X) = 1$, *for the mirror* X^* *should hold*

$$\dim H^1(X^*, T_X) = \dim H^1\left(X^*, \Omega_X^2\right) = h^{2,1}(X^*) = h^{1,1}(X) = 1$$

hence in order to construct the mirror we have to look for a 1*-parameter family.*

It turns out that the right choice is

$$X_\lambda = \left\{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \lambda x_0 x_1 x_2 x_3 x_4 = 0\right\}$$

divided out by the $\mathbb{Z}_5^5$ *action* $(a_0, ..., a_4)(x_0 : ... : x_4) = (\mu^{a_0} x_0 : ... : \mu^{a_4} x_4)$ *with* $\mu = e^{\frac{2\pi i}{5}}$. *Resolving the singularities of this singular quotient without destroying the Calabi-Yau property gives the mirror.*

In constructing mirror pairs we encounter several problems:

1. Even if we start with a manifold, we encounter singular varieties (see quintic in $\mathbb{P}^4$).

   First of all we know, that we can resolve the singularities by a sequence of blowups:

   Theorem 7 (Hironaka) *Let* X *be a normal projective variety over an algebraically closed field of characteristic* 0. *For any proper subvariety* $D \subset X$ *there exists a smooth projective variety* Y *and a birational morphism* $f : Y \to X$ *s.t.* $f^{-1}(D)$ *is a divisor with only simple normal crossings (and* f *is a composition of blowups in smooth closed centers).*

For a proof, and an algorithmic implementation of Hironaka´s theorem see [8].

2. Of course we want the resolved variety to still be a Calabi-Yau:

   Definition 8 *A birational projective morphism $f : Y \to X$ with Y smooth and X at worst Gorenstein canonical singularities is called* ***crepant desingularization*** *of X if $f^*K_X = K_Y$.*

   If the crepant desingularizations of $Y \to X$ resp. $Y^* \to X^*$ exist, we can define a topological mirror pair by

$$h^{p,q}(Y) = h^{d-p,q}(Y^*) \ \forall 0 \leq p, q \leq d$$

   However it is not obvious that this is well defined: If a crepant desingularization exists, it is not necessarily unique. In particular, given two crepant resolutions $Y_1 \to X$ and $Y_2 \to X$ it is not clear a priori, that the Hodge numbers of Y_1 and Y_2 are equal (we will prove that they are indeed equal).

   Example 9 *Let X_0 be a smooth Fano embedded by a very ample line bundle L with $L^k = K_{X_0}^{-l}$ $(k, l \in \mathbb{N})$, let $E = \mathcal{O}_{X_0} \oplus L$ and consider the map*

$$\begin{array}{cccc} f: & Y = \mathbb{P}(E) & \to & X \subset \mathbb{P}\big(\overbrace{H^0(X_0, \mathcal{O}_{X_0} \oplus L)}^{\simeq H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)\right)}\big) \\ & \pi \downarrow\uparrow \sigma & & \\ & X_0 & & \end{array}$$

   which is the contraction of $\sigma(X_0) \simeq X_0$ where $\sigma : X_0 \to \mathbb{P}(E)$ is the section of the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ corresponding to the natural embedding $\mathcal{O}_{X_0} \hookrightarrow \mathcal{O}_{X_0} \oplus L$. Hence $X = C(X_0)$ is a cone over X_0.

   We now calculate the discrepancy: π is the blowup of X in the singular point of X with exceptional locus $D = \sigma(X_0) \simeq X_0$. So

$$\mathcal{O}_Y(D)\mid_D = \mathcal{N}_{D/Y} = L^{-1}$$

   Write

$$K_Y = \pi^* K_X \otimes \mathcal{O}_Y(D)^a$$

   and restrict to D

$$K_Y \mid_D = \mathcal{O}_Y(D)^a \mid_D = L^{-a}$$

The adjunction formula yields

$$L^{-\frac{k}{l}} = K_D = \left(K_Y \otimes \mathcal{O}_Y(D)\right)|_D = L^{-a} \otimes L^{-1} = L^{-(a+1)}$$

so $a = \frac{k}{l} - 1$.

Now consider the case of a smooth quadric $X_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. *Then we can write* $X = S(1,1,0)$ *as*

$$X = \left\{ \det \begin{pmatrix} y_0 & y_2 \\ y_1 & y_3 \end{pmatrix} = 0 \right\} \subset \mathbb{P}^4$$

so $P = (0:0:0:1)$ *is the singular point of* X. *The discrepancy is*

$$K_Y = \pi^* K_X + D$$

We now calculate a small and hence crepant resolution of $X = C(X_0)$. *Let*

$$\begin{aligned} E_1 &:= \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \\ E_2 &:= \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \end{aligned}$$

The maps from $\mathbb{P}(E_1) = \mathbb{P}(E_2)$ *to* $\mathbb{P}\left(H^0\left(\mathbb{P}(E_i), \mathcal{O}_{\mathbb{P}(E_i)}(1)\right)\right) = \mathbb{P}\left(H^0\left(\mathbb{P}^1, E_i\right)\right)$ *give rise to a diagram*

$$\begin{array}{lcl} \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right) & \overset{\sim}{\rightarrow} & S(2,2,1) =: Y_{small} \\ \| & & \\ \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}\right) & \rightarrow & S(1,1,0) = X \end{array}$$

and hence to a morphism $Y_{small} \rightarrow X$, *e.g. with*

$$S(2,2,1) = \left\{ \text{minors}\left(2, \begin{pmatrix} x_0 & x_1 & x_3 & x_4 & x_6 \\ x_1 & x_2 & x_4 & x_5 & x_7 \end{pmatrix}\right) = 0 \right\} \subset \mathbb{P}^7$$

A morphism $g : Y_{small} \rightarrow X$ *is given by*

$$g(x_0 : \ldots : x_7) = (x_0 : x_1 : x_3 : x_4 : x_6)$$

and the exceptional locus is $\mathbb{P}^1$.

3. There are also C-Y varieties, which do not have crepant desingularizations. Nevertheless we want to have a notion of mirror symmetry for these. We will see some examples later.

First idea:
Define stringy Hodge numbers $h_{st}^{p,q}(X)$ for singular varieties. The obvious conditions they should satisfy are:

0. For smooth varieties they should coincide with the usual Hodge numbers.
1. If there exists a crepant desingularization $Y \to X$, they should coincide with the Hodge numbers of Y.
2. Even if there is no crepant desingularization we still want a notion of mirror symmetry.

We will see that for the enlarged class of varieties we will be considering, there in general is no notion of stringy Hodge numbers. But as there is a (not nessecarily polynomial) generating function encoding equivalent information, there is still a notion of mirror symmetry.

2 Mirror Symmetry and Stringy Hodge Numbers

Let X be an irreducible normal algebraic variety of dimension d over $\mathbb{C}$.

2.1 The Hodge weight filtration and the E Polynomial

The cohomology groups $H^k(X,\mathbb{Q})$ carry a natural mixed Hodge structure [5], [6], which is given by the following data:

An increasing filtration

$$0 = W_{-1} \subset W_0 \subset ... \subset W_{2k} = H^k(X,\mathbb{Q})$$

on $H^k(X,\mathbb{Q})$ called weight filtration, and an decreasing filtration

$$H^k(X,\mathbb{C}) = F^0 \supset F^1 \supset ... \supset F^k \supset F^{k+1} = 0$$

on $H^k(X,\mathbb{C}) = H^k(X,\mathbb{Q}) \otimes \mathbb{C}$ called Hodge filtration. We then have

$$H^{p,q}\left(H^k(X,\mathbb{C})\right) = F^p Gr_{p+q} H^k(X,\mathbb{C}) \cap \overline{F^q Gr_{p+q} H^k(X,\mathbb{C})}$$

where

$$\begin{aligned} Gr_l H^k(X,\mathbb{Q}) &:= (W_l / W_{l-1}) \\ F^p Gr_l H^k(X,\mathbb{C}) &:= \mathrm{Im}\left(F^p \cap (W_l \otimes \mathbb{C}) \to Gr_l H^k(X,\mathbb{Q}) \otimes \mathbb{C}\right) \end{aligned}$$

and the filtrations have the property that $F^p Gr_l H^k(X, \mathbb{C})$ gives a (pure) Hodge structure of weight l on $Gr_l H^k(X, \mathbb{Q})$.

We therefore have a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p,q} H^{p,q}\left(H^k(X, \mathbb{C})\right)$$

In [4] one can find a proof, that also the cohomology with compact support $H_c^i(X, \mathbb{Q})$ admits a mixed Hodge structure.

Definition 10 *The E-**polynomial** $E(X; u, v) \in \mathbb{Q}[u, v]$ (coefficients in $\mathbb{Z}$) of a complex normal algebraic variety X of dimension d is then defined as*

$$E(X; u, v) := \sum_{0 \leq p,q \leq d} \sum_{0 \leq i \leq 2d} (-1)^i h^{p,q}\left(H_c^i(X)\right) u^p v^q \tag{2}$$

So we have a map from the category of normal algebraic varieties $\mathcal{V}_{\mathbb{C}}$ to $\mathbb{Q}[u, v]$ by

$$E : ob\mathcal{V}_{\mathbb{C}} \to \mathbb{Q}[u, v], \ X \mapsto E(X; u, v)$$

associating to each X its E polynomial.

Important properties of the E polynomial:

Proposition 11 *Let X and X_i complex normal algebraic variety.*

1. *If $X = \bigcup_i X_i$ is stratified by a disjoint union of locally closed subvarieties then*
$$E(X) = \sum_i E(X_i)$$

2. $$E(X_1 \times X_2) = E(X_1) \cdot E(X_2)$$

3. *If $X \to B$ is a locally trivial fibration and F the fiber over the closed point then*
$$E(X) = E(F) \cdot E(B)$$

A proof can be found in the previously mentioned paper by Danilov and Khovanskii. Note that the number of F_q-points of a variety has similar properties as E.

Remark 12 *For smooth compact X of dimension d*

$$E(X;u,v) := \sum_{0\leq p,q\leq d} h^{p,q}(X)\, u^p v^q \tag{3}$$

with $h^{p,q}(X) = \dim H^{p,q}_{\bar\partial}(X) = \dim H^q(X, \Omega^p_X)$.

- *Hodge duality for X is equivalent to*

$$E(X;u,v) = E(X;v,u)$$

- *Poincare duality for X is equivalent to*

$$E(X;u,v) = (uv)^d E\left(X;u^{-1},v^{-1}\right)$$

- *Mirror symmetry for 2 varieties X and X^* is equivalent to*

$$E(X;u,v) = u^d E\left(X^*;u^{-1},v\right)$$

Remark 13 *Consider a stratification $X = U \cup C$ with X and C compact. The long exact sequence for cohomology with compact support reads as*

$$\ldots \to H^k_c(U) \overset{\varphi_k}{\to} H^k(X) \overset{\psi_k}{\to} H^k(C) \overset{\delta_k}{\to} H^{k+1}_c(U) \to \ldots$$

where φ_k is given by continuation by 0, ψ_k is given by restriction and the boundary map δ_k is given by $\varpi \mapsto d(\beta \cdot r^*\varpi)$ where r is the retract of a tubular neighborhood of C and β is a bump function on this neighborhood.

Example 14 *For $X = \mathbb{P}^1$, $U = \mathbb{C}$ and $C = \{pt\}$ we have*

$k=$	$H^k_c(\mathbb{C})$	$\to$	$H^k(\mathbb{P}^1)$	$\to$	$H^k(pt)$
2	uv		uv		0
1	0		0		0
0	0		1		1

where we denote the Hodge filtration by the corresponding E monomials. The E-polynomials are

$$E(\mathbb{C}) = uv \quad E(\mathbb{P}^1) = 1 + uv \quad E(pt) = 1$$

Remark: The long exact sequence decomposes in short ones if all varieties only have even cohomology.

For $X = \mathbb{P}^3$, C an elliptic curve and $U = \mathbb{P}^3 - C$ we have

$k =$	$H^k_c(U)$	$\rightarrow$	$H^k(\mathbb{P}^3)$	$\rightarrow$	$H^k(C)$
6	$(uv)^3$		$(uv)^3$		0
5	0		0		0
4	$(uv)^2$		$(uv)^2$		0
3	0		0		0
2	$u+v$		uv		uv
1	0		0		$-u-v$
0	0		1		1

The E-polynomials are

$$E(U) = u + v + (uv)^2 + (uv)^3 \quad E(\mathbb{P}^3) = 1 + uv + (uv)^2 + (uv)^3 \quad E(C) = 1 - u - v + uv$$

So a shift in the cohomological weight occurs (Note also the sign of the $u+v$ term).

Example 15 (continued) *The corresponding Hodge filtration for the cohomology of U:*

$$k = 0 \qquad \begin{array}{c|c} \multicolumn{2}{c}{Gr_l H^k} \\ l = & 0 \\ \hline \boxed{F_0} & \end{array}$$

$$0 = W_{-1} \subset W_0 = H^0(U, \mathbb{Q})$$

$$k = 2 \qquad \begin{array}{ccc|ccccc} & & & \multicolumn{5}{c}{Gr_l H^k} \\ & & l = & 0 & 1 & 2 & 3 & 4 \\ \hline F_0 & F_1 & F_2 & & & & & \\ & & & & u & & & \\ & & & & v & & & \end{array}$$

$$0 = W_{-1} = W_0 \subset W_1 = \ldots = W_4 = H^2(U, \mathbb{Q})$$

$$k = 4 \qquad \begin{array}{ccccc|ccccccccc} & & & & & \multicolumn{9}{c}{Gr_l H^k} \\ & & & & l = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline F_0 & F_1 & F_2 & F_3 & F_4 & & & & & & & & & \\ & & & & & & & & & (uv)^2 & & & & \end{array}$$

$$0 = W_{-1} = \ldots = W_3 \subset W_4 = \ldots = W_8 = H^4(U, \mathbb{Q})$$

similar for $k = 6$.

Example 16 *We continue our example (9) of the cone over the quadric calculating the E-functions:*

The cohomology ring $H^*(X_0)$ of X_0 is generated by $h_i = pr_i^* c_1(\mathbb{P}^1)$, $i = 1, 2$ and hence $1, h_1, h_2, h_1h_2$ is a basis as a vector space, so

$$E(X_0) = 1 + 2uv + (uv)^2$$

which agrees with the product formula $E(X_0) = E(\mathbb{P}^1)^2 = (1 + uv)^2$.

$H^*(Y)$ is a free module over $H^*(X_0)$ with basis $1, c = c_1(\mathcal{O}_Y(1))$ and hence $1, c, h_1, h_2, ch_1, ch_2, h_1h_2, ch_1h_2$ is a vector space basis (where h_i is short for π^*h_i), so

$$E(Y) = 1 + 3uv + 3(uv)^2 + (uv)^3$$

$H^*(Y_{small})$ is a free module over $H^*(\mathbb{P}^1)$ with basis $1, c, c^2$ with $c = c_1(\mathcal{O}_{Y_{small}}(1))$ and hence $1, h, c, c^2, ch, hc^2$ is a vector space basis ($h = \pi^*c_1(\mathbb{P}^1)$), so

$$E(Y_{small}) = 1 + 2uv + 2(uv)^2 + (uv)^3$$

So the E polynomials

$$\begin{aligned} E(Y\backslash X_0) &= E(Y) - E(X_0) = \left(1 + 3uv + 3(uv)^2 + (uv)^3\right) - \left(1 + 2uv + (uv)^2\right) \\ &= uv + 2(uv)^2 + (uv)^3 \\ E\left(Y_{small}\backslash \mathbb{P}^1\right) &= E(Y_{small}) - E\left(\mathbb{P}^1\right) = \left(1 + 2uv + 2(uv)^2 + (uv)^3\right) - (1 + uv) \\ &= uv + 2(uv)^2 + (uv)^3 \end{aligned}$$

agree as expected because of $Y\backslash X_0 \cong X\backslash P \cong Y_{small}\backslash \mathbb{P}^1$. By this we can also calculate

$$E(X) = 1 + uv + 2(uv)^2 + (uv)^3$$

2.2 Varieties with Canonical Singularities

Definition 17 *A normal projective variety X is said to have at worst **canonical singularities** if*

- *X is $\mathbb{Q}$-Gorenstein, equivalently $K_X \in Div(X) \otimes \mathbb{Q}$*
- *For a resolution of singularities $f : Y \to X$ s.t. the exceptional locus of f is a divisor E whose irreducible components $D_1, ..., D_r$ are smooth divisors with only simple normal crossings and $K_Y = f^*K_X + \sum_{i=1}^r a_iD_i$, we have*

  $$a_i \geq 0 \text{ for all } i$$

  (the discrepancy divisor is effective).

2.3 The Stringy E-Function

From now on we consider a normal projective d dimensional variety X with at worst Gorenstein canonical singularities, $f : Y \to X$ a resolution of singularities with $D_1, ..., D_r$ the smooth components of the exceptional locus with only simple normal crossings.

Let $I = \{1, ..., r\}$ and set for any $J \subset I$

$$D_J = Y \cap \bigcap_{j \in J} D_j$$

$$D_J^\circ = D_J \backslash \bigcup_{i \in I \backslash J} D_i$$

This gives a stratification $D_J = \bigcup_{J', J \subset J'} D_{J'}^\circ$.

Definition 18 *We define the* ***stringy*** E***-function*** E_{st} *of* X *as*

$$E_{st}(X; u, v) := \sum_{J \subset I} E\left(D_J^\circ; u, v\right) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \tag{4}$$

Remark 19 *If* X *is Gorenstein, then the* $a_j \in \mathbb{Z}_{\geq 0}$ *and hence* $E_{st}(X; u, v) \in \mathbb{Z}[[u, v]] \cap \mathbb{Q}(u, v)$*.* $E_{st}(X; u, v)$ *is not a rational function in general.*

Now we will state the main theorem, assuring that $E_{st}(X; u, v)$ is well defined. We will prove this theorem by motivic integration:

Theorem 20 $E_{st}(X; u, v)$ *does not depend on the resolution* $f : Y \to X$*, in particular* $E_{st}(X; u, v)$ *is well defined.*

As a direct Corollary, we have:

Corollary 21 *If* X *is smooth* $E_{st}(X; u, v) = E(X; u, v)$*.*

Remark 22 *Let´s first make an easy observation:* E_{st} *is not affected by the blowup* $f : Y \to X$ *of a point* P *in smooth* X*: The exceptional locus of* f *is* $D_1 = \mathbb{P}^{d-1}$ *and the discrepancy is*

$$K_Y = f^* K_X + (d - 1) D_1$$

$$\begin{aligned}
E_{st}(X) &= E(Y \backslash D_1) + E(D_1) \frac{uv - 1}{(uv)^{a_1+1} - 1} = E(Y \backslash D_1) + E\left(\mathbb{P}^{d-1}\right) \frac{uv - 1}{(uv)^d - 1} \\
&= E(Y \backslash D_1) + \left(1 + uv + ... + (uv)^{d-1}\right) \frac{uv - 1}{(uv)^d - 1} \\
&= E(Y \backslash D_1) + 1 = E(X)
\end{aligned}$$

The idea of the proof of (20) is the following:

The universality of the map $[-]$ from the category of complex algebraic to the Grothendieck ring gives a factorization of $E : ob\mathcal{V}_{\mathbb{C}} \to \mathbb{Q}[u,v]$ through the Grothendieck ring $\mathcal{M}$

$$\begin{array}{ccc} ob\mathcal{V}_{\mathbb{C}} & \xrightarrow{E} & \mathbb{Q}[u,v] \\ [-]\searrow & & \nearrow E \\ & \mathcal{M} & \end{array}$$

The goal is to write

$$E_{st}(X;u,v) = \sum_{J\subset I} E\left(D_J^\circ;u,v\right) \prod_{j\in J} \frac{uv-1}{(uv)^{a_j+1}-1} = E\left(\int_{J_\infty(Y)} F_D d\mu \mathbb{L}^d\right)$$

for a suitable function F_D associated to the discrepancy divisor ($J_\infty(Y)$ the bundle of formal arcs on Y), after extending E to $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ (where $\mathbb{L} = [\mathbb{C}]$) and to an appropriate completion. The transformation rule then gives, that the motivic integral does not depend on the resolution.

2.4 Stringy Hodge numbers

In order to prove Poincare Duality we need the following lemma, which proves useful also in a more general sense:

Lemma 23 *For all $J \subset I$*

$$E\left(D_J^\circ;u,v\right) = \sum_{J' \text{ with } J\subset J'} (-1)^{|J|-|J'|} E\left(D_J;u,v\right)$$

so

$$E_{st}(X;u,v) = \sum_{J\subset I} E\left(D_J;u,v\right) \prod_{j\in J}\left(\frac{uv-1}{(uv)^{a_j+1}-1} - 1\right)$$

Instead of proving this (easy) we will illustrate it in an example:

Example 24 *$I = \{1,2\}$ and $a_1 = 1$, $a_2 = 1$ and $D_1 \cap D_2 = \{P\}$:*

Stratification by D_J°:

$$Y = (Y \setminus (D_1 \cup D_2)) \cup (D_1 \setminus \{P\}) \cup (D_2 \setminus \{P\}) \cup \{P\}$$

The first formula reads:

$$\begin{aligned} E\left(D_\emptyset^\circ\right) &= E\left(Y\setminus(D_1\cup D_2)\right) = E(Y) - E(D_1) - E(D_2) + E(P) \\ E\left(D_{\{j\}}^\circ\right) &= E\left(D_j \setminus P\right) = E(D_j) - E(P) \\ E\left(D_{\{1,2\}}^\circ\right) &= E(P) = E(P) \end{aligned}$$

By this

$$\begin{aligned}
E_{st}\left(X;u,v\right) &= \sum_{J\subset I} E\left(D_J^{\circ};u,v\right)\prod_{j\in J}\frac{uv-1}{\left(uv\right)^{a_j+1}-1} = \\
&= \left(E\left(Y\right)-E\left(D_1\right)-E\left(D_2\right)+E\left(P\right)\right) \\
&\quad + \left(E\left(D_1\right)-E\left(P\right)\right)\frac{1}{uv+1} + \left(E\left(D_2\right)-E\left(P\right)\right)\frac{1}{uv+1} \\
&\quad + E\left(P\right)\frac{1}{\left(uv+1\right)^2} \\
&= E\left(Y\right) \\
&\quad + E\left(D_1\right)\left(\frac{1}{uv+1}-1\right) + E\left(D_2\right)\left(\frac{1}{uv+1}-1\right) \\
&\quad + E\left(P\right) - 2E\left(P\right)\frac{1}{uv+1} + E\left(P\right)\frac{1}{\left(uv+1\right)^2}
\end{aligned}$$

Theorem 25 (Poincare Duality) *$E_{st}\left(X;u,v\right)$ has the following properties:*

$$\begin{aligned}
E_{st}\left(X;u,v\right) &= \left(uv\right)^d E_{st}\left(X;u^{-1},v^{-1}\right) \\
E_{st}\left(X;0,0\right) &= 1
\end{aligned}$$

Proof. From the Lemma (23) we have

$$\begin{aligned}
E_{st}\left(X;u,v\right) &= \sum_{J\subset I} E\left(D_J;u,v\right)\prod_{j\in J}\left(\frac{uv-1}{\left(uv\right)^{a_j+1}-1}-1\right) \qquad (5) \\
&= \sum_{J\subset I} E\left(D_J;u,v\right)\prod_{j\in J}\left(\frac{uv-\left(uv\right)^{a_j+1}}{\left(uv\right)^{a_j+1}-1}\right)
\end{aligned}$$

We can check duality for each term separately: Poincare Duality holds for each closed D_J

$$\left(uv\right)^{d-|J|} E\left(D_J;u^{-1},v^{-1}\right) = E\left(D_J;u,v\right)$$

and

$$\prod_{j\in J}\left(\frac{uv-\left(uv\right)^{a_j+1}}{\left(uv\right)^{a_j+1}-1}\right) = \left(uv\right)^{|J|}\prod_{j\in J}\left(\frac{\left(uv\right)^{-1}-\left(uv\right)^{-a_j-1}}{\left(uv\right)^{-a_j-1}-1}\right)$$

Substituting $u=v=0$ in the equality (5) yields

$$E_{st}\left(X;0,0\right) = \sum_{J\subset I} E\left(D_J;0,0\right)\prod_{j\in J}\left(\frac{-1}{-1}-1\right) = E\left(Y;0,0\right)$$

Definition 26 *If E is a polynomial, then* $\deg(E) = 2d$. *We then define the* ***stringy Hodge numbers*** *of X as*

$$h_{st}^{p,q}(X) = (-1)^{p+q}\, coeff\,(E_{st}, u^p v^q)$$

So $h_{st}^{p,q}(X) = 0$ outside the Hodge diamond and $h_{st}^{0,0}(X) = h_{st}^{d,d}(X) = 1$.

2.5 Crepant Resolutions and Mirror Symmetry

Theorem 27 *If X admits a crepant resolution $f : Y \to X$ then $E_{st}(X; u, v) = E(Y; u, v)$.*

Proof. By Hironaka´s theorem there is a smooth Z and a birational morphism $g : Z \to Y$, s.t. $f \circ g$ is a resolution of singularities of X and the exceptional locus has normal crossing irreducible components $D_1', ..., D_{r'}'$. Let $D_1, ..., D_r$ be the irreducible normal crossing components of g. Write the discrepancy loci as

$$K_Z = g^* K_Y + \sum_{i=1}^{r} a_i D_i$$

and for $f \circ g$ as

$$K_Z = g^* f^* K_X + \sum_{i=1}^{r'} a_i' D_i'$$

By $f^* K_X = K_Y$ we have

$$\sum_{i=1}^{r} a_i D_i = \sum_{i=1}^{r'} a_i' D_i'$$

Denote the supports of the exceptional loci of $f \circ g$ and f by $\operatorname{supp} D$ and $\operatorname{supp} D'$. Clearly $\operatorname{supp} D \subset \operatorname{supp} D'$, so if $D_i' \subset \operatorname{supp} D' \setminus \operatorname{supp} D$ then $a_i' = 0$.

So computing $E_{st}(Y)$ from f and $E_{st}(X)$ from $f \circ g$ gives the same formula, since terms with $a_i = 0$ don´t contribute. Hence

$$E(Y) = E_{st}(Y) = E_{st}(X)$$

since Y is smooth.

Remark 28 *In particular $E_{st}(X; u, v)$ is polynomial and hence the stringy Hodge numbers of X exist. Equivalently, if $E_{st}(X; u, v)$ is not polynomial, then X admits no crepant resolution.*

Definition 29 *Two Calabi-Yau varieties X and X^* are called **topological mirror pair**, if their stringy E functions satisfy*

$$E_{st}\left(X;u,v\right)=u^{d}E_{st}\left(X^{*};u^{-1},v\right)$$

Remark 30 *This is well defined even in the case, when E_{st} is not polynomial. If a crepant resolution exists, the notion of a stringy topological mirror pair coincides with the previous definition.*

Example 31 *Now we return to the example (9) and (16) from above: $X_0 \subset \mathbb{P}^d$ a smooth quadric and*

$$f:\mathbb{P}\left(\mathcal{O}_{X_0}\left(1\right)\oplus\mathcal{O}_{X_0}\right)=Y\rightarrow X=C\left(X_0\right)$$

with discrepancy divisor $D_1 \simeq X_0$.

For $d=3$ we had $X=S\left(1,1,0\right)$, we computed a small resolution

$$S\left(2,2,1\right)=Y_{small}\rightarrow X$$

and calculated

$$\begin{aligned}E\left(Y_{small}\right)&=1+2uv+2\left(uv\right)^2+\left(uv\right)^3\\E\left(Y\right)&=1+3uv+3\left(uv\right)^2+\left(uv\right)^3\\E\left(D_1\right)&=1+2uv+\left(uv\right)^2\\E\left(Y\backslash D_1\right)&=uv+2\left(uv\right)^2+\left(uv\right)^3\end{aligned}$$

So the stringy E function E_{st} is

$$\begin{aligned}E_{st}\left(X\right)&=E\left(D_{\emptyset}^{\circ}\right)+E\left(D_{\{1\}}^{\circ}\right)\frac{uv-1}{\left(uv\right)^2-1}=E\left(Y\backslash D_1\right)+E\left(D_1\right)\frac{1}{uv+1}\\&=\left(uv+2\left(uv\right)^2+\left(uv\right)^3\right)+\left(1+uv\right)^2\frac{1}{uv+1}\\&=1+2uv+2\left(uv\right)^2+\left(uv\right)^3=E\left(Y_{small}\right)\end{aligned}$$

and, as predicted by the theorem (27), the stringy Hodge Numbers of X indeed coincide with the Hodge numbers of the small resolution.

Exercise 32 *Show that in the preceding example E_{st} is not a polynomial for $d>3$ (in particular X does not admit a crepant resolution).*

3 Motivic Integration

In the following let Y be a algebraic complex manifold of dimension d.

3.1 Overview

Our to goal is to prove the theorem (20) (in the case of X Gorenstein). Short overview of the objects involved in the proof:

- The main part of the proof will take place on the smooth variety Y. The necessary data is the discrepancy divisor on D on Y, but this can also be any other effective divisor with simple normal crossings.
- We will use the following objects:
  - The bundle $J_\infty(Y)$ of formal arcs in Y. Think of the fiber over $y \in Y$ as all formal curves in Y through y i.e. power series expansions in one variable in Y.
  - A function $F_D : J_\infty(Y) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ associating to each $\gamma_y \in J_\infty(Y)$ its intersection multiplicity with D, encoding all the information about D.
  - The Grothendieck ring $\mathcal{M}$ of isomorphy classes of algebraic varieties.
  - A measure μ on cylinder sets in $J_\infty(Y)$ taking values in $\mathcal{M}[\mathbb{L}^{-1}]$ where $\mathbb{L}$ denotes the class of $\mathbb{C}$.
  - The level sets $F_D^{-1}(s)$, which will turn out to be cylinder sets for finite s.
  - A completion R of $\mathcal{M}[\mathbb{L}^{-1}]$ and an extended measure taking values in R. We will need this, because the level set $F_D^{-1}(\infty)$ is not a cylinder set.
  - The motivic integral

    $$\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu := \sum_{s \in \mathbb{Z}_{\geq 0} \cup \{\infty\}} \mu\left(F_D^{-1}(s)\right) \cdot \mathbb{L}^{-s}$$

    which we will compute explicitly.
- The universality of the map $[-]$ from the category of complex algebraic varieties to the Grothendieck ring gives a factorization of $E : \mathcal{V}_\mathbb{C} \to \mathbb{Q}[u,v]$ through the Grothendieck ring

$$\begin{array}{ccc} ob\mathcal{V}_\mathbb{C} & \xrightarrow{E} & \mathbb{Q}[u,v] \\ {[-]} \searrow & & \nearrow E \\ & \mathcal{M} & \end{array}$$

which extends to R. It turns out that the whole theory is constructed in a way that

$$E_{st}\left(X;u,v\right)=E\left(\int_{J_\infty(Y)}F_D d\mu\mathbb{L}^d\right)$$

and that the motivic integral for the discrepancy divisor does not depend on the choice of the resolution, because of the transformation rule.

- As a corollary to the transformation rule we will also prove, that birational smooth Calabi-Yau have the same Hodge numbers.

3.2 The bundle of formal arcs

Definition 33 *Let $y\in Y$.*

*A **k-jet** over y is a morphism*

$$\gamma_y : Spec\mathbb{C}\left[z\right]/\left\langle z^{k+1}\right\rangle\rightarrow Y$$
$$\text{with } \gamma_y\left(Spec\mathbb{C}\right)=y$$

(Note that $Spec\mathbb{C}\left[z\right]/\left\langle z^{k+1}\right\rangle$ has only one element, which is the ideal $\langle z\rangle$, and this corresponds to $Spec\mathbb{C}$). In local coordinates of Y think of a k-jet as a d-tuple of polynomials of degree k whose constant terms are zero.

*A **formal arc** over y is a morphism*

$$\gamma_y : Spec\mathbb{C}\left[[z]\right]\rightarrow Y$$
$$\text{with } \gamma_y\left(Spec\mathbb{C}\right)=y$$

In local coordinates of X a formal arc is a d-tuple of power series whose constant terms are zero.

We can build bundles $J_k\left(Y\right)\rightarrow Y$ and $J_\infty\left(Y\right)\rightarrow Y$ with fibers $\left(J_k\left(Y\right)\right)_y=\{k\text{-jets of } Y \text{ over } y\}$and $\left(J_\infty\left(Y\right)\right)_y=\{\text{formal arcs of } Y \text{ over } y\}$ over y.

Example 34 *First explore this in the case $Y=\mathbb{A}^d=Spec\left(\mathbb{C}\left[x_1,....,x_d\right]\right)$.*

We have to consider the ring homomorphisms

$$\mathbb{C}\left[x_1,....,x_d\right]\rightarrow\mathbb{C}\left[z\right]/\left\langle z^{k+1}\right\rangle$$

which are given by classes of polynomials

$$x_i\mapsto a_{i0}+a_{i1}\bar{z}+...+a_{ik}\bar{z}^k$$

so $J_k(Y)$ is parametrized by the coefficients a_{ij} $i = 1, ..., d$, $j = 0, ..., k$ hence

$$J_k(Y) \cong \mathbb{A}^{d\cdot(k+1)}$$

(Remark: In general $J_k(Y)$ is a bundle. Here a factor $\mathbb{A}^d$ corresponds to the base.).

One can show that this stays true if Y is smooth.

Let $y \in Y = V(f_1, ..., f_s) \subset \mathbb{A}^n$ and $d = \dim Y$. Because Y is smooth after a change of coordinates we can by the implicit function theorem for power series find unique power series

$$y_{d+1}, ..., y_n \in \langle x_1, ..., x_d \rangle \cdot \mathbb{C}[[x_1, ..., x_d]]$$

s.t.

$$f_i(x_1, ..., x_d, y_{d+1}, ..., y_n) = 0 \text{ for all } i = 1, ..., s$$

and $(x_1, ..., x_d, y_{d+1}(x_1, ..., x_d), ..., y_n(x_1, ..., x_d))$ parametrizes Y in some open set in a topology, where the implicit function theorem holds. Now specifying power series $X_1, ..., X_d \in \mathbb{C}[[z]]$ is the same as specifying an arc in Y:

$$\begin{aligned} X_1 &= X_1(z) \\ &\vdots \\ X_d &= X_1(z) \\ X_{d+1} &= Y_{d+1}(X_1(z), ..., X_d(z)) \\ &\vdots \\ X_n &= Y_n(X_1(z), ..., X_d(z)) \end{aligned}$$

This stays of course also true for any truncation of the power series to $\mathbb{C}[z] / \langle z^{k+1} \rangle$.

So the coefficients of the X_i give the locally trivial bundle structure of $J_k(Y)$ over an open set in the etale topology (where the implicit function theorem holds). Then one has to show, that this induces a locally trivial bundle structure in the Zariski topology.

Proposition 35 *Let Y be smooth of dimension d. There are bundles $J_k(Y) \to Y$ resp. $J_\infty(Y) \to Y$ with fibers over y the k-jets resp. arcs of Y over y.*

$J_k(Y)$ is an $\mathbb{A}^{k\cdot d}$ bundle over Y, in particular it is smooth (this is no longer true if Y is not smooth, as the fiber dimension may jump).

$J_s(Y)$ is an $\mathbb{A}^{(s-r)\cdot d}$ bundle over $J_k(Y)$ for $s > r$.

For each k the natural truncation $\mathbb{C}[[z]] \to \mathbb{C}[z] / \langle z^{k+1} \rangle$ induces a bundle map

$$\pi_k : J_\infty(Y) \to J_k(Y)$$

which is surjective, because every $\gamma_y \in J_k(Y)$ can be lifted to $J_\infty(Y)$.

Similarly the truncation induces surjective bundle maps

$$\pi_{s,k} : J_s(Y) \to J_k(Y)$$

for $s > k$.

The standard proof goes as follows: Define the $J_k(Y)$ as schemes, show:

Lemma 36 *If $Z \to Y$ is etale, then $J_k(Z) \cong J_k(Y) \times_Y Z$*

So since a smooth Y is locally etale over $\mathbb{A}^n$ in the Zariski topology, we get the assertion.

Remark 37
- *The k-valued points of $J_k(Y)$ are the $\mathbb{C}[z] / \langle z^{k+1} \rangle$ valued points of Y.*
- $J_0(Y) \cong Y$
- *$J_1(Y) \cong T_Y$ is the tangent bundle. In general we can describe $J_k(Y)$ by k-th. order derivations.*
- *The inverse limit of the $J_k(Y)$ is $J_\infty(Y) = \lim\limits_{\leftarrow} J_k(Y)$.*

3.3 Constructible sets

Definition 38 *A subset of a variety is called **constructible**, if it is a finite disjoint union of locally closed subvarieties (with respect to the Zariski topology).*

*A subset $C \subset J_\infty(Y)$ is called a **cylinder set**, if $C = \pi_k^{-1}(B_k)$ with $B_k \subset J_k(Y)$ constructible. B_k is called k-**basis** of C.*

Remark 39 *If $\pi_k(C)$ is a k-basis for C, then for all $s > k$*

$$\pi_s(C) = \pi_{s,k}^{-1}(\pi_k(C))$$

is the s-base of C, i.e. $\pi_s(C)$ is constructible and $C = \pi_s^{-1}\pi_s(C)$.

Finite unions and intersections and complements of cylinder sets are cylinder sets.

Proof. Easy exercise, e.g. for the union of two cylinder sets: Let $C = \pi_k^{-1}(B_k)$ with $B_k \subset J_k(Y)$ and $C' = \pi_s^{-1}(B'_s)$ with $B'_s \subset J_s(Y)$ and suppose $k < s$. Then with $B_s := \pi_{s,k}^{-1}(B_k)$ we have $C = \pi_s^{-1}(B_s)$ and $C \cup C' = \pi_s^{-1}(B_s \cup B'_s)$.

3.4 The function F_D encoding the information of D

In the following let $D = \sum_{i=1}^{r} a_i D_i$ be an effective divisor on Y and $I = \{1, ..., r\}$.

Definition 40 *For $y \in Y$ a point and g a local equation of D in a neighborhood U of y. For an arc γ_y over y we call*

$$\gamma_y \cdot D := \textit{vanishing order of } g\left(\gamma_y\left(z\right)\right) \textit{ at } z = 0$$

the intersection number of γ_y and D. Let F_D be the function

$$\begin{array}{rll} F_D : & J_\infty\left(Y\right) & \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma_y & \mapsto \gamma_y \cdot D \end{array}$$

The goal is to integrate $F_D : J_\infty\left(Y\right) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $\gamma_y \mapsto \gamma_y \cdot D$ over $J_\infty\left(Y\right)$. In order to do this we will show, that the level sets are cylinder sets and compute $\mu\left(F_D^{-1}\left(s\right)\right)$ with respect to some measure μ.

3.5 Local description of the level sets of F_D

With the same notation as in the case of D being an exceptional locus, write for any $J \subset I$

$$D_J = Y \cap \bigcap_{j \in J} D_j \quad D_J^\circ = D_J \backslash \bigcup_{i \in I \backslash J} D_i$$

The partition $Y = \dot{\bigcup\limits_{J \subset I}} D_J^\circ$ induces a partition

$$J_\infty\left(Y\right) = \dot{\bigcup_{J \subset I}} \pi_0^{-1}\left(D_J^\circ\right)$$

So we can produce a partition of any subset of $J_\infty\left(Y\right)$ by intersecting with the $\pi_0^{-1}\left(D_J^\circ\right)$.

We will do this for the level sets $F_D^{-1}\left(s\right)$ by the following construction.

Remark 41 *It holds*

$$F_D = \sum_{i=1}^{r} a_i F_{D_i}$$

hence for $\gamma_y \in J_\infty\left(Y\right)$ we have $F_D\left(\gamma_y\right) = \sum_{i=1}^{r} a_i F_{D_i}\left(\gamma_y\right)$ and for the terms of the sum the following holds:

Remark 42 *For $\gamma_y \in J_\infty(Y)$*

$$F_{D_i}(\gamma_y) = 0 \Leftrightarrow y \notin D_i$$

Combining both and writing $s = F_D(\gamma_y)$, $m_i = F_{D_i}(\gamma_y)$ it holds $\sum_{i=1}^r a_i m_i = s$ and

$$(m_j > 0 \Leftrightarrow j \in J) \Leftrightarrow (y \in \bigcap_{j\in J} D_j \text{ and } y \notin \bigcap_{j\in I\setminus J} D_j) \Leftrightarrow y \in D_J^\circ$$

So with the

Definition 43 *For $J \subset I$ and $s \in \mathbb{Z}_{\geq 0}$*

$$M_{J,s} := \left\{(m_1, ..., m_r) \in \mathbb{Z}_{\geq 0}^r \mid a_1 m_1 + ... + a_r m_r = s \text{ and } m_j > 0 \Leftrightarrow j \in J\right\}$$

we proved $\gamma_y \in \pi_0^{-1}(D_J^\circ) \cap F_D^{-1}(s) \Leftrightarrow (F_{D_1}(\gamma_y), ..., F_{D_r}(\gamma_y)) \in M_{J,s}$ that is:

Proposition 44 *For each $J \subset I$ it holds*

$$\pi_0^{-1}(D_J^\circ) \cap F_D^{-1}(s) = \dot{\bigcup_{m\in M_{J,s}}} \bigcap_{i\in I} F_{D_i}^{-1}(m_i) \tag{6}$$

and hence for each $s \in \mathbb{Z}_{\geq 0}$

$$F_D^{-1}(s) = \dot{\bigcup_{J\subset I}} \dot{\bigcup_{m\in M_{J,s}}} \left(\bigcap_{i\in I} F_{D_i}^{-1}(m_i)\right) \tag{7}$$

is a finite partition of the level set $F_D^{-1}(s)$.

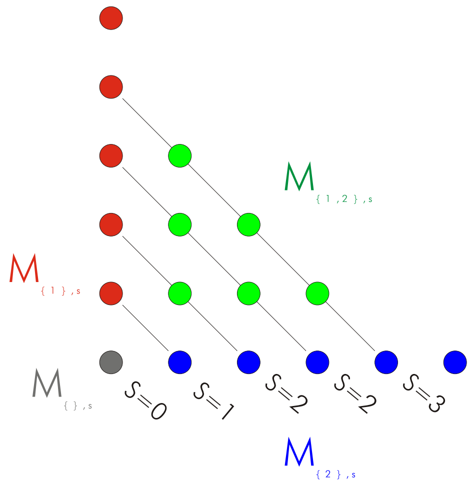
Example 45 *We describe this in an example:*

$$D = D_1 + D_2$$

with $D_1 \cap D_2 = \{P\}$. Set for J

$$M_{J,s} = \left\{(m_1, m_2) \in \mathbb{Z}_{\geq 0}^2 \mid m_1 + m_2 = s \text{ and } m_j > 0 \Leftrightarrow j \rightarrow J\right\}$$

which gives an partition of $\mathbb{Z}^2_{\geq 0}$:



So

$$\pi_0^{-1}\left(D^\circ_\emptyset\right) \cap F_D^{-1}(s) = \left\{ \begin{array}{ll} F_{D_1}^{-1}(0) \cap F_{D_2}^{-1}(0) & \textit{for } s = 0 \\ \emptyset & \textit{otherwise} \end{array} \right\}$$

$$\pi_0^{-1}\left(D^\circ_{\{1\}}\right) \cap F_D^{-1}(s) = F_{D_1}^{-1}(m_1) \cap F_{D_2}^{-1}(0)$$

$$\pi_0^{-1}\left(D^\circ_{\{1,2\}}\right) \cap F_D^{-1}(s) = \bigcup_{\substack{m_1+m_s=s \\ m_1,m_2\geq 1}} F_{D_1}^{-1}(m_1) \cap F_{D_2}^{-1}(m_2)$$

Hence in this case

$$F_D^{-1}(s) = \bigcup_{\substack{m_1+m_s=s \\ m_1,m_2\geq 0}} F_{D_1}^{-1}(m_1) \cap F_{D_2}^{-1}(m_2)$$

Remark 46 (Local description of the level sets) *Let $D = \sum_{i=1}^r a_i D_i$ be an effective, normal crossing divisor. There is a finite covering of Y by coordinate charts $U \to V \subset \mathbb{C}^d$, s.t. D is given in the chart by the equation $z_1^{a_1} \cdot \ldots \cdot z_{b_y}^{a_{b_y}} = 0$ with $a_1, \ldots, a_{b_y} \geq 1$ and $0 \leq b_y \leq d$ (without loss of generality). For $s \in \mathbb{Z}_{\geq 0}$ and $(m_1, \ldots, m_r) \in M_{J,s}$ write*

$$U_{m_1,\ldots,m_r} := \bigcap_{i \in I} F_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U)$$

- *If J is not contained in the set of variables appearing in the local equation of D i.e. $J \not\subset \{1, \ldots, b_y\}$, then $U \cap D_J^\circ = \emptyset$ hence*

$$U_{m_1,\ldots,m_r} = \emptyset$$

- *Otherwise i.e. if* $J \subset \{1, ..., b_y\}$,

  we can describe $\bigcap_{i \in I} F_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U)$ *(i.e. power series with vanishing order* m_i*.on* D_i*) as the preimage by* π_k *of the power series truncated to degree*

$$k = \max\{m_1, ..., m_r \mid j = 1, ..., r\}$$

  So we get $d - |J|$ *power series with vanishing constant term*

$$0 + \overset{\in \mathbb{C}}{(...)} z + ... + \overset{\in \mathbb{C}}{(...)} z^k$$

  and $|J|$ *power series of the form*

$$0 + ... + 0 + \overset{\in \mathbb{C}^*}{(...)} z^{m_j} + \overset{\in \mathbb{C}}{(...)} z^{m_j+1} + ... + \overset{\in \mathbb{C}}{(...)} z^k$$

  and conclude with (6)

$$\begin{aligned} U_{m_1,...,m_r} &\simeq \pi_k^{-1}\left((U \cap D_J^\circ) \times \mathbb{C}^{k(d-|J|)} \times (\mathbb{C}^*)^{|J|} \times \mathbb{C}^{\sum_{j \in J} k - m_j}\right) \qquad (8) \\ &= \pi_k^{-1}\left((U \cap D_J^\circ) \times \mathbb{C}^{d \cdot k - \sum_{j \in J} m_j} \times (\mathbb{C}^*)^{|J|}\right) \end{aligned}$$

Example 47 *For* $U = \mathbb{C}^2$ *and* $D = D_1 + D_2$ *given by* $z_1 z_2 = 0$, $P = 0 \in \mathbb{C}^2$ *we have*

		$F_{D_1}^{-1}(m_1) \cap F_{D_2}^{-1}(m_2) \cap \pi_0^{-1}(U) \simeq$
$J = \emptyset$	$(m_1, m_2) = (0, 0)$	$\pi_0^{-1}(\mathbb{C}^2 \backslash (D_1 \cup D_2))$
$J = \{1\}$	$(m_1, m_2) = (m_1, 0)$	$\pi_{m_1}^{-1}((D_1 \backslash \{P\}) \times \mathbb{C}^{m_1} \times \mathbb{C}^*)$
$J = \{1, 2\}$	$(m_1, m_2), m_1 \geq m_2 \geq 1$	$\pi_{m_1}^{-1}\left(\{P\} \times \mathbb{C}^{m_1 - m_2} \times (\mathbb{C}^*)^2\right)$

With the remark (46) about the local description of the level sets it easily follows:

Proposition 48 (Finite level sets are cylinder sets) *For a normal crossing effective divisor* D *the level set* $F_D^{-1}(s)$ *is a cylinder set for all* $s \in \mathbb{Z}_{\geq 0}$.

$F_D^{-1}(\infty)$ *is a countable intersection of cylinder sets.*

Proof. By (8) $\bigcap_{i \in I} F_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U)$ is a cylinder set and hence also $\bigcap_{i \in I} F_{D_i}^{-1}(m_i)$ as a finite union of cylinder sets and so by (7) also $F_D^{-1}(s)$ again as a finite union of cylinder sets is a cylinder set.

For the infinity level set $F_D^{-1}(\infty)$, we remark that

$$F_D^{-1}(\infty) = \bigcap_{k \in \mathbb{Z}_{\geq 0}} \pi_k^{-1} \pi_k \left(F_D^{-1}(\infty)\right)$$

We only have to check this for each D_i and here it holds because a power series is 0 if and only if all truncations to any order are 0.

3.6 The Grothendieck ring and the measure for cylinder sets in $J_\infty(Y)$

Definition 49 *The* ***Grothendieck ring of complex algebraic varieties*** $\mathcal{M}$ *is the free abelian group of isomorphism classes of complex algebraic varieties modulo the subgroup generated by* $[X]-[V]-[X-V]$ *for closed subsets* $V \subset X$*, with a ring structure given by*

$$[X]\cdot[X'] = [X \times X']$$

We call the neutral element $[point] =: 1$ *and* $[\mathbb{C}] =: \mathbb{L}$.

(So $[\mathbb{C}^*] = \mathbb{L} - 1$).

Remark 50 *Let* $f : X \to B$ *be a locally trivial fibration (with respect to Zariski topology) of complex algebraic varieties and* F *the fiber of* f *over a closed point, then*

$$[X] = [F]\cdot[B]$$

Proof. There is a finite open covering $\{U_i\}$ of B s.t. X is trivial over U_i. By induction on the number of U_i we only need to prove this for a covering $U_1 \cup U_2$: We then have $f^{-1}(U_1) \simeq F \times U_1$ so

$$\left[f^{-1}(U_1)\right] = [F]\cdot[U_1]$$

and as $B\backslash U_1 \subset U_2$ we also have $f^{-1}(B\backslash U_1) \simeq F \times (B\backslash U_1)$

$$\left[f^{-1}(B\backslash U_1)\right] = [F]\cdot[B\backslash U_1] = [F]\cdot([B]-[U_1])$$

where the last equation holds because $B\backslash U_1$ is closed. Since $f^{-1}(B\backslash U_1)$ is closed in X we get

$$[X] = \left[f^{-1}(U_1)\right] + \left[f^{-1}(B\backslash U_1)\right] = [F]\cdot([U_1]+[B]-[U_1]) = [F]\cdot[B]$$

Definition 51 *For a cylinder set* $C = \pi_k^{-1}(B_k)$ *with* $B_k \subset J_k(Y)$ *we define an additive measure*

$$\begin{aligned} \mu : \{\textit{cylinder sets}\} &\to \mathcal{M}\left[\mathbb{L}^{-1}\right] \\ \mu(C) &:= [B_k]\cdot\mathbb{L}^{-d(k+1)} \end{aligned} \tag{9}$$

Proof. μ is well defined: Let $s > k$ and $C = \pi_k^{-1}(B_k)$ a cylinder. Then with $B_s := \pi_{s,k}^{-1}(B_k)$ we have $C = \pi_s^{-1}(B_s)$. As the fibers of $\pi_{s,k}$ are $\mathbb{C}^{d\cdot(s-k)}$ by the remark about locally trivial fibrations we have $[B_s] = \left[\mathbb{C}^{d\cdot(s-k)} \times B_k\right]$ hence

$$[B_s] \cdot \mathbb{L}^{-d(s+1)} = \mathbb{L}^{d(s-k)} \cdot [B_k] \cdot \mathbb{L}^{-d(s+1)} = [B_k] \cdot \mathbb{L}^{-d(k+1)}$$

A similar argument proves the additivity.

By this definition we can compute the measure of the level sets $F_D^{-1}(s)$ for $s \in \mathbb{Z}_{\geq 0}$, but not of $F_D^{-1}(\infty)$. Hence we will have to define an additive measure with values in a completion R of $\mathcal{M}$ s.t. it is compatible with μ and $F_D^{-1}(\infty)$ has measure 0. We will denote the new measure again by μ.

3.7 Measure of the level sets

Proposition 52 (Measure of the level sets) *For $s \in \mathbb{Z}_{\geq 0}$ the measure of the level sets $F_D^{-1}(s)$ of F_D is*

$$\mu\left(F_D^{-1}(s)\right) = \sum_{J \subset I} \sum_{m \in M_{J,s}} [D_J^\circ] \cdot \mathbb{L}^{-\sum_{j\in J} m_j} \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-d}$$

Proof. As above we cover Y by coordinate charts $U_j \to V \subset \mathbb{C}^d$, s.t. D is given in each chart by an equation $z_1^{a_1} \cdot ... \cdot z_{b_y}^{b_y} = 0$. Then for $s \in \mathbb{Z}_{\geq 0}$ and $(m_1, ..., m_r) \in M_{J,s}$

$$\bigcap_{i\in I} F_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U_j) = \pi_k^{-1}\left((U_j \cap D_J^\circ) \times \mathbb{C}^{d\cdot k - \sum_{j\in J} m_j} \times (\mathbb{C}^*)^{|J|}\right)$$

or empty. So

$$\bigcap_{i\in I} F_{D_i}^{-1}(m_i) = \pi_k^{-1}(C) \text{ with}$$

$$[C] = \left[D_J^\circ \times \mathbb{C}^{d\cdot k - \sum_{j\in J} m_j} \times (\mathbb{C}^*)^{|J|}\right] = [D_J^\circ] \cdot \mathbb{L}^{d\cdot k - \sum_{j\in J} m_j} \cdot (\mathbb{L}-1)^{|J|}$$

hence

$$\mu\left(\bigcap_{i\in I} F_{D_i}^{-1}(m_i)\right) = [D_J^\circ] \cdot \mathbb{L}^{-\sum_{j\in J} m_j} \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-d}$$

and with the disjoint partition (7) the claim follows.

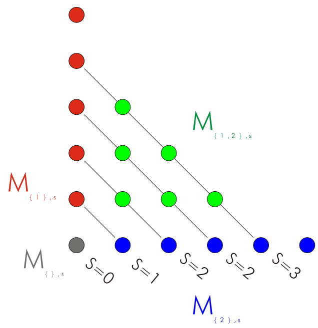
Example 53 *In the previous example $D = D_1 + D_2$ (i.e. $a_1 = a_2 = 1$) with $D_1 \cap D_2 = \{P\}$ and $I = \{1, 2\}$. So*

$$\mu\left(F_D^{-1}(0)\right) = \mathbb{L}^{-d} \cdot [Y \setminus (D_1 \cup D_2)]$$

and for $s \geq 1$:

$$\mu\left(F_D^{-1}(s)\right) = \mathbb{L}^{-d} \cdot ($$
$$[D_1 \backslash P] \cdot \mathbb{L}^{-s} \cdot (\mathbb{L}-1) \quad + \quad (s-1) \cdot [P] \cdot \mathbb{L}^{-s} \cdot (\mathbb{L}-1)^2 \quad + \quad [D_2 \backslash P] \cdot \mathbb{L}^{-s} \cdot (\mathbb{L}-1))$$

corresponding to the diagonals in the picture



3.8 The completion of $\mathcal{M}\left[\mathbb{L}^{-1}\right]$ and calculation of the motivic integral

Remark 54 *The map $[-] : ob\mathcal{V}_{\mathbb{C}} \to \mathcal{M}$ is the universal map being additive on disjoint unions of constructible sets and multiplicative on products, so any other map $E : ob\mathcal{V}_{\mathbb{C}} \to R$ with the same properties factors through $[-]$. So the universality of $[-]$ gives a factorization of $E : \mathcal{V}_{\mathbb{C}} \to \mathbb{Q}[u,v]$ through the Grothendieck ring*

$$\begin{array}{ccc} ob\mathcal{V}_{\mathbb{C}} & \xrightarrow{E} & \mathbb{Q}[u,v] \\ [-] \searrow & & \nearrow E \\ & \mathcal{M} & \end{array}$$

(which we will also denominate by E), so $E(\mathbb{L}) = E([\mathbb{C}]) = E(\mathbb{C}) = uv$.

E can be extended by $E(\mathbb{L}^{-1}) := (uv)^{-1}$ to a map

$$E : \mathcal{M}\left[\mathbb{L}^{-1}\right] \to \mathbb{Q}\left[u, v, (uv)^{-1}\right]$$

Remark 55 *Let´s first do the calculations to see the critical steps and then give the completion R, where these calculations are well defined:*

Our goal is to write the following expression (which in the case of resolutions of singularities is the stringy E-function E_{st}) as

$$\sum_{J \subset I} E\left(D_J^\circ; u, v\right) \prod_{j \in J} \frac{uv-1}{(uv)^{a_j+1}-1} = E\left(\sum_{J \subset I} \left[D_J^\circ\right] \prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1}\right)$$

and then express the argument of E as an integral by

$$\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu := \sum_{s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}} \mu\left(F_D^{-1}(s)\right)\cdot\mathbb{L}^{-s} = \sum_{J\subset I}[D_J^\circ]\prod_{j\in J}\frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1}\mathbb{L}^{-d}$$

Check this calculation, using Proprosition (52):

$$\begin{aligned}
\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu &= \sum_{s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}} \mu\left(F_D^{-1}(s)\right)\cdot\mathbb{L}^{-s} \qquad (10)\\
&= \sum_{s\in\mathbb{Z}_{\geq 0}}\sum_{J\subset I}\sum_{m\in M_{J,s}}[D_J^\circ]\cdot\mathbb{L}^{-\sum_{j\in J}m_j}\cdot(\mathbb{L}-1)^{|J|}\cdot\mathbb{L}^{-d}\cdot\mathbb{L}^{-s}\\
&= \sum_{s\in\mathbb{Z}_{\geq 0}}\sum_{J\subset I}\sum_{m\in M_{J,s}}[D_J^\circ]\cdot(\mathbb{L}-1)^{|J|}\cdot\mathbb{L}^{-\sum_{j\in J}(a_j+1)m_j}\cdot\mathbb{L}^{-d}\\
&= \sum_{J\subset I}[D_J^\circ]\cdot(\mathbb{L}-1)^{|J|}\cdot\sum_{s\in\mathbb{Z}_{\geq 0}}\sum_{m\in M_{J,s}}\left(\prod_{j\in J}\mathbb{L}^{-(a_j+1)m_j}\right)\cdot\mathbb{L}^{-d}\\
&= \sum_{J\subset I}[D_J^\circ]\cdot(\mathbb{L}-1)^{|J|}\cdot\prod_{j\in J}\left(\sum_{m_j\geq 1}\mathbb{L}^{-(a_j+1)m_j}\right)\cdot\mathbb{L}^{-d}\\
&= \sum_{J\subset I}[D_J^\circ]\cdot(\mathbb{L}-1)^{|J|}\cdot\prod_{j\in J}\frac{1}{\mathbb{L}^{a_j+1}-1}\cdot\mathbb{L}^{-d}\\
&= \sum_{J\subset I}[D_J^\circ]\cdot\prod_{j\in J}\frac{(\mathbb{L}-1)}{\mathbb{L}^{a_j+1}-1}\cdot\mathbb{L}^{-d}
\end{aligned}$$

So we want to define a measure for a countable union of disjoint cylinder sets by

$$\mu\left(\dot{\bigcup_{j\in\mathbb{Z}_{\geq 0}}} C_j\right) = \sum_{j\in\mathbb{Z}_{\geq 0}}\mu(C_j)$$

but countable sums are not defined in $\mathcal{M}[\mathbb{L}^{-1}]$ so we have to pass to some completion and also have to show, that the limit does not depend on the order of summation.

Definition 56 *We consider the completion R of $\mathcal{M}[\mathbb{L}^{-1}]$ with respect to the filtration*

$$F^m := F^m\mathcal{M}\left[\mathbb{L}^{-1}\right] = \left\langle [V]\,\mathbb{L}^{-j} \mid j\geq \dim V + m\right\rangle \text{ for } m\in\mathbb{Z}$$

(so $F^{m+1}\subset F^m$) and denote the completion map by

$$\phi : \mathcal{M}\left[\mathbb{L}^{-1}\right]\to R$$

The completion behaves somehow analogous to formal Laurent series.

Remark 57 *The completion is constructed in the following way: For $a_n \in \mathcal{M}\left[\mathbb{L}^{-1}\right]$ we define*

$$\lim_{n\to\infty} a_n = 0 \quad \Leftrightarrow \quad \forall \varepsilon \in \mathbb{Z} \quad \exists n_0 \in \mathbb{N} : a_n \in F^{\varepsilon} \quad \forall n \geq n_0$$

and

$$a_n \text{ is a Cauchy sequence} \quad \Leftrightarrow \quad \forall \varepsilon \quad \exists n_0 : a_n - a_m \in F^{\varepsilon} \quad \forall n, m \geq n_0$$

Two Cauchy sequences a_n and b_n are called equivalent, if $a_n - b_n$ is a null sequence (i.e. $\lim_{n\to\infty} a_n - b_n = 0$) and the set of equivalence classes is the completion R. We have a well defined addition on R by $(a_n)+(b_n) = (a_n + b_n)$ and multiplication $(a_n) \cdot (b_n) = (a_n \cdot b_n)$.

The completion map $\phi(a) = (a)$ sending a to the constant sequence (a) is a ring homomorphism.

By definition a sequence $(a_n) \subset \mathcal{M}\left[\mathbb{L}^{-1}\right]$ converges with limit value in R iff it is a Cauchy sequence. The limit value is the sequence by itself.

For all $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$

$$F^{\varepsilon_1} \cdot F^{\varepsilon_2} \subset F^{\varepsilon_1+\varepsilon_2}$$

Let´s first check that $(a_n + b_n)$ and $(a_n \cdot b_n)$ are well defined:

- If a_n is a Cauchy sequence then a_n is bounded i.e. $\exists \varepsilon \in \mathbb{Z}$ s.t. $a_n \in F^{\varepsilon}$ $\forall n \in \mathbb{N}$:

  $\exists n_0 : a_n - a_{n_0} \in F^0$ $\forall n \geq n_0$ and $\exists \varepsilon_1$ s.t. $a_0, ..., a_{n_0} \in F^{\varepsilon_1}$ then $a_n \in F^{\min\{0,\varepsilon_1\}}$ $\forall n \geq n_0$.

- For all $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$ we have $F^{\varepsilon_1} \cdot F^{\varepsilon_2} \subset F^{\varepsilon_1+\varepsilon_2}$:

  Let $C_1 \in F^{\varepsilon_1}$ and $C_2 \in F^{\varepsilon_2}$, i.e. $C_i = [V_i] \cdot \mathbb{L}^{-j_i}$ with $j_i \geq \dim V_i + \varepsilon_i$ so $C_1 \cdot C_2 = [V_1 \times V_2] \cdot \mathbb{L}^{-j}$ and

  $$j = j_1 + j_2 \geq \dim V_1 + \dim V_2 + \varepsilon_1 + \varepsilon_2 = \dim(V_1 \times V_2) + \varepsilon_1 + \varepsilon_2$$

  hence $C_1 \cdot C_2 \in F^{\varepsilon_1+\varepsilon_2}$.

- If a_n and b_n are Cauchy sequences then $(a_n + b_n)$ and $(a_n \cdot b_n)$ are again Cauchy sequences:

  $\forall \varepsilon \quad \exists n_0 : a_n - a_m \in F^{\varepsilon} \quad \forall n, m \geq n_0$ and equally n_1 for b_n. Then for the sum

  $$a_n + b_n - (a_m + b_m) = \overbrace{(a_n - a_m)}^{\in F^{\varepsilon}} - \overbrace{(b_n - b_m)}^{\in F^{\varepsilon}} \in F^{\varepsilon} \quad \forall n, m \geq N = \max\{n_0, n_1\}$$

For the product choose a fixed ε_1 s.t. $a_n, b_n \in F^{\varepsilon_1} \forall n$. Then

$$a_n b_n - a_m b_m = \overset{\in F^{\varepsilon_1}}{a_n} \overbrace{(b_n - b_m)}^{\in F^{\varepsilon}} + \overset{\in F^{\varepsilon_1}}{a_n} \overbrace{(a_n - a_m)}^{\in F^{\varepsilon}} \in F^{\varepsilon+\varepsilon_1} \quad \forall n, m \geq N = \max\{n_0, n_1\}$$

Remark: Be aware that the ε_1 can be negative. By choosing N big enough we can absorb the ε_1.

- If c_n and d_n are null sequences, then $c_n + d_n$ is a null sequence.

  $\forall \varepsilon \quad \exists n_0 : c_n \in F^{\varepsilon}$ and $\exists n_1 : d_n \in F^{\varepsilon}$. Then $c_n + d_n \in F^{\varepsilon}$ $\forall n \geq N = \max\{n_0, n_1\}$ and $c_n \cdot d_n \in F^{2\varepsilon} \subset F^{\varepsilon}$ $\forall n \geq N = \max\{n_0, n_1\}$.

- If a_n is a Cauchy sequence and c_n is a null sequence then $a_n c_n$ is a null sequence:

  $\exists \varepsilon_1$ s.t. $a_n \in F^{\varepsilon_1}$ $\forall n$. So $\forall \varepsilon \quad \exists n_0 : c_n \in F^{\varepsilon}$ hence $a_n c_n \in F^{\varepsilon+\varepsilon_1}$.

- If a_n, b_n are Cauchy sequences and c_n, d_n null sequences, then

$$(a_n + c_n) + (b_n + d_n) = (a_n + b_n) + \overbrace{(c_n + d_n)}^{\text{is null sequence}}$$
$$(a_n + c_n) \cdot (b_n + d_n) = a_n b_n + \overbrace{c_n b_n + a_n d_n + c_n d_n}^{\text{is null sequence}}$$

Remark 58 *For sequences $a_n, b_n \in \mathcal{M}\left[\mathbb{L}^{-1}\right]$ it holds:*

1. $\lim_{n\to\infty} a_n = 0 \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} a_n$ *convergent.*
2. *If $\sum_{n=1}^{\infty} a_n$ is convergent and $\sigma : \mathbb{N} \to \mathbb{N}$ bijective, then $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is convergent and has the same limit value.*
3. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then*

$$\left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right) = \sum_{s=1}^{\infty} \sum_{n+m=s} a_n b_m$$

1. $\forall \varepsilon \quad \exists n_0 : a_n \in F^{\varepsilon} \quad \forall n \geq n_0$ so $\sum_{n=n_1}^{n_2} a_n \in F^{\varepsilon} \quad \forall n_1, n_2 \geq n_0$ as F^{ε} is a group, hence the partial sum is a Cauchy sequence.

   On the other hand if $\forall \varepsilon \in \mathbb{Z} \quad \exists n_0 \in \mathbb{N} : \sum_{n=n_1}^{n_2} a_n \in F^{\varepsilon} \quad \forall n_1, n_2 \geq n_0$ then in particular for $n_1 = n_2 = n_0$ we have $a_{n_0} \in F^{\varepsilon}$.

2. Cauchy sequence + null sequence is a Cauchy sequence. So we have to show, that this series differs from the first one only by a null sequence: $\forall\varepsilon \quad \exists n_0 : a_n \in F^{\varepsilon} \quad \forall N \geq n_0$. Take n_1 s.t. $\{\sigma(0), ..., \sigma(n_1)\} \supset \{0, 1, ..., n_0 - 1\}$. Then

$$\overbrace{\sum_{n=1}^{m} a_{\sigma(n)} - \sum_{n=1}^{n_0-1} a_n}^{\in F^{\varepsilon}} - \overbrace{\sum_{n=n_0}^{m} a_n}^{\in F^{\varepsilon}} \in F^{\varepsilon} \text{ for all } m \geq n_1$$

so the difference is indeed a null sequence.

3. $\forall\varepsilon \quad \exists n_0 : a_n \in F^{\varepsilon} \quad \forall n \geq n_0$ and similar n_1 for (b_n). Hence

$$D_N := \left(\sum_{n=1}^{N} a_n\right) \cdot \left(\sum_{n=1}^{N} b_n\right) - \sum_{s=1}^{N} \sum_{n+m=s} a_n b_n = \sum_{\substack{n,m \leq N \\ n+m > N}} a_n b_n \in F^{\varepsilon} \quad \forall N \geq \max\{n_0, n_1\}$$

so $\lim_{N\to\infty} D_N = 0$ i.e. the two sequences only differ by a null sequence.

Definition 59 *So we can define a measure on*

$$\left\{ \dot{\bigcup_{j\in\mathbb{Z}_{\geq 0}}} C_j \mid C_j \text{ disjoint cylinder sets with } \mu(C_j) \to 0 \text{ for } j \to \infty \right\}$$

by

$$\mu\left(\dot{\bigcup_{j\in\mathbb{Z}_{\geq 0}}} C_j\right) = \sum_{j\in\mathbb{Z}_{\geq 0}} \mu(C_j) \in R$$

In the completion R the limit exists (because $\mu(C_j)$ is a null sequence) and it is independent of the order of summation.

Definition 60 *The **motivic integral** of the effective normal crossing divisor D on the complex manifold Y is defined as:*

$$\int_{J_\infty(Y)} F_D d\mu := \int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu := \sum_{s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}} \mu\left(F_D^{-1}(s)\right) \cdot \mathbb{L}^{-s}$$

Proposition 61 *If D is an effective normal crossing divisor then F_D is measurable.*

Proof. We will see later, that $\mu\left(F_D^{-1}(\infty)\right) = 0$, so we only have to show, that $\mu\left(F_D^{-1}(s)\right) \cdot \mathbb{L}^{-s}$ is a null sequence. Actually even $\mu\left(F_D^{-1}(s)\right)$ is a null sequence:

$$\mu\left(F_D^{-1}(s)\right) = \sum_{J \subset I} \sum_{m \in M_{J,s}} [D_J^\circ] \cdot \mathbb{L}^{-\sum_{j \in J} m_j} \cdot (\mathbb{L}-1)^{|J|} \cdot \mathbb{L}^{-d}$$

For all $J \subset I$ and $m \in M_{J,s}$

$$\dim\left([D_J^\circ] \cdot (\mathbb{L}-1)^{|J|}\right) = d - |J| + |J| = d$$

hence

$$\left([D_J^\circ] \cdot (\mathbb{L}-1)^{|J|}\right) \cdot \mathbb{L}^{-\sum_{j \in J} m_j - d} \in F^{\sum_{j \in J} m_j + d - d} = F^{\sum_{j \in J} m_j}$$

So $\mu\left(F_D^{-1}(s)\right)$ is a null sequence, because $\sum_{j \in J} m_j \to \infty$ as $s \to \infty$ if $D \neq 0$ and if $D = 0$ then

$$\mu\left(F_D^{-1}(s)\right) = \left\{ \begin{array}{ll} [Y] \cdot \mathbb{L}^{-d} & \text{for } s = 0 \\ 0 & \text{else} \end{array} \right\}$$

Now our claim follows with:

Lemma 62 $\mu\left(F_D^{-1}(\infty)\right) = 0$.

Proof. Omitted.

Theorem 63 *The motivic integral then evaluates as*

$$\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu = \sum_{J \subset I} [D_J^\circ] \prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1} \mathbb{L}^{-d}$$

in particular

$$\int_{J_\infty(Y)} \mathbb{L}^{-F_0} d\mu = [Y] \cdot \mathbb{L}^{-d}$$

So the motivic integral takes values in the subring

$$R' = \phi\left(\mathcal{M}\left[\mathbb{L}^{-1}\right]\right)\left[\left\{\frac{1}{\mathbb{L}^j - 1}\right\}_{j \in \mathbb{N}}\right] \subset R$$

where we should think of $\left(\mathbb{L}^j - 1\right)^{-1}$ as the corresponding power series expansion.

Remark 64 *Omitting the* $\mathbb{L}^{-s}$ *factor in the sum leads to*

$$\sum_{s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}} \mu\left(F_D^{-1}(s)\right) = \sum_{J\subset I} [D_J^\circ]\,\mathbb{L}^{-d} = [Y]\,\mathbb{L}^{-d} = \int_{J_\infty(Y)} \mathbb{L}^{-F_0} d\mu$$

for any effective normal crossing divisor $D \neq 0$ *(see calculation (10)). Of course the left hand side is* $\mu(J_\infty(Y))$ *and the right hand side can be interpreted as* $\int_{J_\infty(Y)} 1 d\mu$.

Proposition 65 *The map*

$$E : \mathcal{M}\left[\mathbb{L}^{-1}\right] \to \mathbb{Q}\left[u, v, (uv)^{-1}\right]$$

can be uniquely extended to a map

$$E : R' \to \mathbb{Q}\left[u, v, (uv)^{-1}, \left\{\frac{1}{(uv)^j - 1}\right\}_{j\in\mathbb{N}}\right]$$

by $E\left(\frac{1}{\mathbb{L}^j-1}\right) = \frac{1}{(uv)^j-1}$.

Proof. The kernel of the completion map

$$\ker\phi = \bigcap_{m\in\mathbb{Z}} F^m$$

is annihilated by E: For $a \in F^m$ it holds $\deg E(a) \leq -2m$, so for $a \in \ker\phi$, we have $\deg E(a) = -\infty$ i.e. $E(a) = 0$. Hence E factors through $\phi\left(\mathcal{M}\left[\mathbb{L}^{-1}\right]\right)$. Remark: It is unknown, whether $\ker\phi = 0$.

So by the previous calculation (10):

Remark 66 *We have*

$$\sum_{J\subset I} E\left(D_J^\circ; u, v\right) \prod_{j\in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} = E\left(\sum_{J\subset I} [D_J^\circ] \prod_{j\in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j+1} - 1}\right) = E\left(\int_{J_\infty(Y)} \mathbb{L}^{-F_D} d\mu \mathbb{L}^d\right)$$

This in particular holds for the discrepancy divisor D *of a resolution of singularities* $Y \to X$ *and gives* $E_{st}(X)$ *in this case.*

3.9 The transformation rule for the motivic integral

Theorem 67 (Transformation rule) *Let X and Y be smooth and $f : X \to Y$ a proper birational morphism and $W = K_Y - f^*K_X$ the discrepancy divisor ($W = (\det Jac_f) \geq 0$). Then*

$$\int_{J_\infty(Y)} F_D d\mu = \int_{J_\infty(X)} F_{f^*D+W} d\mu \tag{11}$$

Proof. We will only prove this for $D = 0$. The general proof is analogous.

For the sequence $D_s = F_W^{-1}(s)$ it holds $\mu(D_s) \to 0$ as $s \to \infty$ (see proof of the existence of the motivic integral). For $k \in \mathbb{Z} \cup \{\infty\}$ we have maps

$$\begin{array}{llll} f_k : & J_k(X) & \to & J_k(Y) \\ & \gamma_x & \mapsto & f \circ \gamma_x \in J_k(Y)_{f(x)} \end{array}$$

which are bijective outside the set of zero measure $F_W^{-1}(\infty)$, hence the sequence $C_s = f_\infty(D_s)$ is a partition of $J_\infty(Y)$ up to a set of measure 0. By a lemma (see below) we know that C_k is again a cylinder set and $\mu(D_s) = \mu(C_s) \cdot \mathbb{L}^s$ in particular $\mu(C_s) = \mu(D_s) \cdot \mathbb{L}^{-s} \to 0$. Hence $\sum_{s\in\mathbb{Z}} \mu(C_s)$ exists and equals $\mu(J_\infty(Y)) = [Y] = \int_{J_\infty(Y)} F_0 d\mu$. Putting all together we get

$$\int_{J_\infty(Y)} F_0 d\mu = \mu(J_\infty(Y)) = \sum_{s\in\mathbb{Z}} \mu(C_s) = \sum_{s\in\mathbb{Z}} \mu(D_s)\mathbb{L}^{-s} = \int_{J_\infty(X)} F_W d\mu$$

Lemma 68 *For $D_s = F_W^{-1}(s)$ the set $C_s = f_\infty(D_s)$ is again a cylinder set and*

$$\mu(D_s) = \mu(C_s) \cdot \mathbb{L}^s$$

Proof. As shown above D_s is a cylinder set. So there is a k and a constructible set B s.t. B is a k-basis of D_s. $f_k(B)$ is again constructible (proof omitted) and for $k \geq s$ the following diagram commutes

$$\begin{array}{ccccccc} D_s & \subset & J_\infty(X) & \xrightarrow{f_\infty} & J_\infty(Y) & \supset & f_\infty(D_s) = C_s \\ & & \pi_k \downarrow & & \pi_k \downarrow & & \\ B & \subset & J_k(X) & \xrightarrow{f_k} & J_k(Y) & \supset & f_k(B) = \pi_k(f_\infty(D_s)) \end{array}$$

We want to show that $f_k|_B : B \to f_k(B)$ is bundle with fiber $\mathbb{A}^s$.

Let $\psi_y \in \pi_k\left(f_\infty\left(D_s\right)\right)$ i.e. $\psi_y = \pi_k\left(f \circ \gamma_x\right)$ with $\gamma_x \in D_s$. Locally ψ_y is given by a d-tuple of polynomials of degree k, so writing the coefficient vectors in the rows we get

$$\psi_y = \begin{pmatrix} a_{1,0} & \cdots & a_{1,k} & 0 & \cdots & \\ \vdots & & \vdots & \vdots & & \\ a_{d,0} & \cdots & a_{d,k} & 0 & \cdots & \end{pmatrix}$$

and its preimage by π_k is

$$\begin{aligned}\pi_k^{-1}\left(\psi_y\right) &= \left\{\psi_y + t^{k+1}v \mid v \in \mathbb{C}\left[[t]\right]^d\right\} \\ &= \left\{f\left(\gamma_x\right) + t^{k+1}v \mid v \in \mathbb{C}\left[[t]\right]^d\right\} \\ &= \begin{pmatrix} a_{1,0} & \cdots & a_{1,k} & * & * & \cdots \\ \vdots & & \vdots & \vdots & & \\ a_{d,0} & \cdots & a_{d,k} & * & * & \cdots \end{pmatrix}\end{aligned}$$

After a change of coordinates over $\mathbb{C}\left[[t]\right]$ we can assume, that the Jacobian matrix $J_f\left(\psi_y\right)$ has the form

$$Jac_f\left(\psi_y\right) = \begin{pmatrix} t^{s_1} & & \\ & \ddots & \\ & & t^{s_d} \end{pmatrix}$$

with $s_1 + ... + s_d = s$.

After Taylor expansion of f for $u \in \mathbb{C}\left[[t]\right]^d$

$$f\left(\gamma_x + u\right) = f \circ \gamma_x + Jac_f\left(\gamma_x\right)u + ...$$

hence $f\left(\gamma_x + u\right) \in \pi_k^{-1}\left(\psi_y\right)$ iff $ord\left(t^{s_j}u_j\right) > k$ i.e.

$$u = \begin{pmatrix} 0 & \cdots & 0 & u_{1,k+1-s_1} & \cdots & \cdots & \cdots & u_{1,k} & \cdots \\ \vdots & & & & & & & & \\ 0 & \cdots & \cdots & \cdots & 0 & u_{d,k+1-s_d} & \cdots & u_{d,k} & \cdots \end{pmatrix}$$

so

$$f_\infty^{-1}\left(\pi_k^{-1}\left(\psi_y\right)\right) = \left\{\gamma_x + w \mid w \in \mathbb{C}\left[[t]\right]^d,\ ord\left(w_i\right) \geq k - s_i \text{ for } i = 1, ..., d\right\}$$

Projecting to $J_k\left(X\right)$ we get

$$\pi_k\left(f_\infty^{-1}\left(\pi_k^{-1}\left(\psi_y\right)\right)\right) = \left\{\gamma_x + w \mid w \in \mathbb{C}\left[t\right]^d,\ ord\left(w_i\right) \geq k - s_i,\ \deg w_i \leq k \text{ for } i = 1, ..., d\right\}$$

i.e.

$$w = \begin{pmatrix} 0 & \cdots & 0 & u_{1,k+1-s_1} & \cdots & \cdots & & \cdots & u_{1,k} & 0 & \cdots \\ \vdots & & & & & & & & & & \\ 0 & \cdots & \cdots & \cdots & & 0 & u_{d,k+1-s_d} & \cdots & u_{d,k} & 0 & \cdots \end{pmatrix}$$

and hence the fiber of π_k in an affine space of dimension $\sum_{i=1}^{d} s_i = s$.

The etale bundle structure lifts again to a locally trivial bundle structure in the Zariski topology.

4 Proof of the independence of E_{st} from the resolution

From now on let X be a normal projective d dimensional variety with at worst Gorenstein canonical singularities, $f : Y \to X$ a resolution of singularities for which the discrepancy divisor $D = K_Y - f^*K_X = \sum_{i=1}^{r} a_i D_i$ has smooth components $D_1, ..., D_r$ and only simple normal crossings. Let $I = \{1, ..., r\}$.

Theorem 69 (Independence of the motivic integral from the resolution) *Let $f_i : Y_i \to X$, $i = 1, 2$ be two resolutions of X with discrepancies D_i. Then*

$$\int_{J_\infty(Y_1)} F_{D_1} d\mu = \int_{J_\infty(Y_2)} F_{D_2} d\mu$$

In particular $E_{st}(X; u, v) = E\left(\int_{J_\infty(Y)} F_D d\mu \cdot \mathbb{L}^d\right)$ is independent of the choice of resolution. The proof can be generalized to the case of X being $\mathbb{Q}$-Gorenstein.

Proof. Let $h : Y \to Y_1 \times_X Y_2$ be a resolution (by Hironaka´s theorem) of the fiber product of Y_1 and Y_2 over X.

$$\begin{array}{ccccc} & & Y & & \\ & & h \downarrow & & \\ & & Y_1 \times_X Y_2 & & \\ & pr_1 \swarrow & & \searrow pr_2 & \\ Y_1 & & & & Y_2 \\ & f_1 \searrow & & \swarrow f_2 & \\ & & X & & \end{array}$$

Then we have proper birational morphisms $h_i = pr_i \circ h : Y \to Y_i$ and $f_0 := f_i \circ pr_i \circ h : Y \to X$ (independent of i because of the fiber product). Denote the discrepancy of f_0 by D_0.

In order to apply the transformation rule to lift the integral from Y_i to Y, we compute the discrepancies of the h_i. Since

$$\begin{aligned} K_Y &= f_0^* K_X + D_0 = h_i^* f_i^* K_X + D_0 = h_i^* \left(K_{Y_i} - D_i\right) + D_0 \\ &= h_i^* K_{Y_i} + \left(D_0 - h_i^* D_i\right) \end{aligned}$$

the discrepancies of the h_i are

$$K_Y - h_i^* K_{Y_i} = D_0 - h_i^* D_i$$

By the transformation rule (11) we get

$$\int_{J_\infty(Y_i)} F_{D_i} d\mu = \int_{J_\infty(Y)} F_{h_i^* D_i + \left(D_0 - h_i^* D_i\right)} d\mu = \int_{J_\infty(Y)} F_{D_0} d\mu$$

independent of i.

5 Birational Calabi-Yau manifolds have equal Hodge numbers

Theorem 70 *Birational Calabi-Yau manifolds have equal Hodge numbers.*

Proof. Let X_1 and X_2 smooth Calabi-Yau manifolds and $f : X_1 \dashrightarrow X_2$ a birational map. Let $d = \dim X_1 = \dim X_2$. Factor f through the resolution Y of the graph:

$$\begin{array}{ccc} & Y & \\ f_1 \swarrow & & \searrow f_2 \\ X_1 & \dashrightarrow & X_2 \end{array}$$

with morphisms f_1 and f_2. Denote by W_i the discrepancy of f_i, so because $K_{X_i} = \mathcal{O}_{X_i}$

$$W := W_1 = K_Y - f_1^* K_{X_1} = K_Y = K_Y - f_2^* K_{X_2} = W_2$$

With the remark about the motivic integral of F_0 and the transformation rule (11) we get

$$[X_i] = \int_{J_\infty(X_i)} F_0 d\mu \cdot \mathbb{L}^d = \int_{J_\infty(Y)} F_{f_i^* 0 + W_i} d\mu \cdot \mathbb{L}^d = \int_{J_\infty(Y)} F_W d\mu \cdot \mathbb{L}^d$$

so $[X_1] = [X_2]$, hence

$$E\left(X_1\right) = E\left(\left[X_1\right]\right) = E\left(\left[X_2\right]\right) = E\left(X_2\right)$$

Remark 71 *Actually we proved, that $[X_1] = [X_2]$, i.e. X_1 and X_2 represent the same class in R.*

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