Massively parallel methods in computer algebra

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SYMBOLIC TOOLS

Outline

- Determining smoothness of algebraic varieties

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- Infrastructure for massively parallel computations: GPI-Space
- SINGULAR and GPI-Space

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- More applications in
  - geometric invariant theory
  - tropical geometry.

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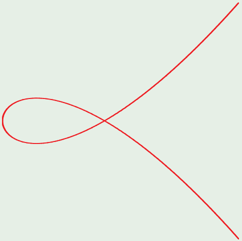

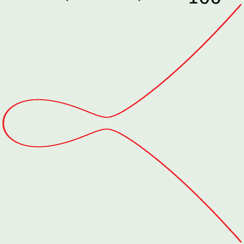

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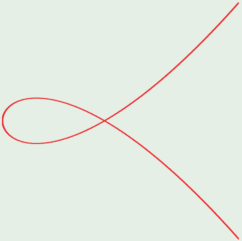

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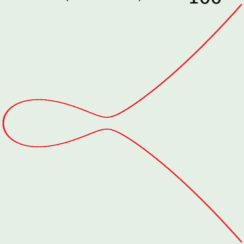

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according to the genus formula $p_g(C) = \frac{(d-1)(d-2)}{2} - \sum_{P \in C} \delta_P(C)$.

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be the **Jacobian matrix***. Then* $\operatorname{rank}(\mathcal{J} \bmod P) \leq c$ *and R_P is a regular local ring iff*

$$\operatorname{rank}(\mathcal{J} \bmod P) = c.$$

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Example

$$I = \langle y^2 - x^2(x+1) \rangle \qquad I = \langle y^2 - x^2(x+1) + \tfrac{1}{100} \rangle$$

$$\mathrm{Sing}(R) = V(\langle x, y \rangle) \qquad \mathrm{Sing}(R) = V(\langle 1 \rangle) = \emptyset$$

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Lemma (Hironaka, 1964)

If $f_1, \ldots, f_s$ is a minimal standard basis of $I\mathcal{O}_{\mathbb{A}^n,p}$, sorted by increasing order, then

$$\nu^*(X,p) = (\mathrm{ord}_{\mathfrak{m}_p}(f_1), \ldots, \mathrm{ord}_{\mathfrak{m}_p}(f_s))$$

depends only on $\mathcal{O}_{X,p}$, and X is singular at p if and only if

$$\nu^*(X,p) >_{lex} (\underbrace{1, \ldots, 1}_{\mathrm{codim}(X)}).$$

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Lemma

If W is a smooth complete intersection of codim s, $X = V(I)$ and $\operatorname{ord}_p(I) \geq 2$ then

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hence $X = V(I)$ is not smooth at p.

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Lemma

If $\{p \in X \mid \mathrm{ord}_p(I) \geq 2\} = \emptyset$ then there is $f \in I$ defining locally in a Zariski neighborhood of p a smooth hypersurface $X \subset Z \subset W$.

Algorithmic smoothness criterion

Lemma

If M is a $s \times s$-submatrix of the Jacobian matrix of W with $\det(M) \neq 0$, *then the x_i not used for differentiation in M induce by translation a local system of parameters $X_{p,j}$ at every point of $p \in W \cap D(\det(M))$. Write $\partial f_i / \partial X_{p,j}$ for the partial derivatives defined in terms of the Cohen structure theorem isomorphism*

$$K[[y_1, \ldots, y_{n-s}]] \cong \widehat{\mathcal{O}_{W,p}}$$

The partials $\partial f_i / \partial X_{p,j}$ can be represented for all $p \in D(\det(M))$ by fixed elements $H_{i,j} \in \mathcal{O}_W(D(\det(M)))$, and we write

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$\{p \in X \mid \mathrm{ord}_p(I) \geq 2\}$ *is defined in* $X \cap D(\det(M))$ *by*

$$I + \langle \partial f_i / \partial X_j \mid i, j \rangle$$

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- Find a set L of $s \times s$ submatrices M of $\mathcal{J}(W)$ with $\det(M) \neq 0$ and

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- Find A such that $A \cdot M = \det(M) \cdot E_s$ and let

$$F := \begin{pmatrix} A & 0 \\ 0 & \det(M) \cdot E_{r-s} \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$$

Algorithmic smoothness criterion

- Then the locus of order ≥ 2 in $D(q)$ is empty iff

$$q \in \sqrt{\langle f_1, \ldots, f_r, \partial F_i / \partial X_j \mid i, j > s \rangle}$$

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- Iteratively obtain tree of charts.

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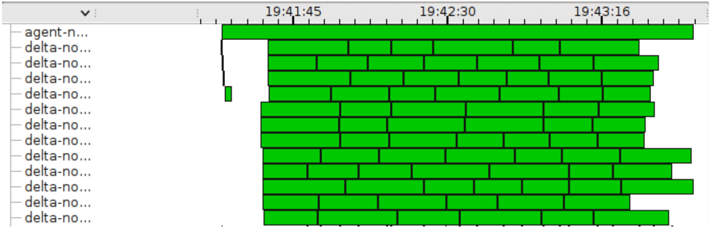
- task based workflow management system for massively parallel computations



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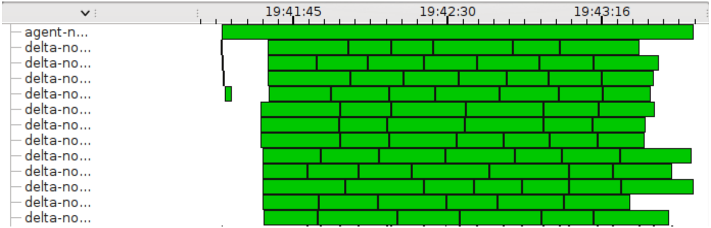

- based on idea of separating coordination and computation.

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Philosophy and Features of GPI-Space

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- Complex coordination hidden from domain experts: automatic parallelization, cost optimized data transfers hiding latency, adaptation to dynamic changes in the computing environment.

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- to switch between different types of memory without changing the implementation.

Structure and Implementation of GPI-Space

Three main components:

- Distributed, scalable, resilient **runtime system** for dynamic computational environments (from small to huge): manages computational resources and memory. Scheduler assigns activities to resources.

GPI-Space: Scheduler

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- **Virtual memory manager**: allows algorithmic building blocks to communicate, share partial results. Communication managed by the runtime system rather than the domain applications.
- Petri net based **workflow engine**: manages the full application state and is responsible for automatic parallelization and dependency tracking.

Petri nets

Introduced by Carl Adam Petri (1926–2010) in 1962 to describe concurrent asynchronous systems (this is how real world physics works!). In fact he invented them already much earlier to remember chemical reactions in school.

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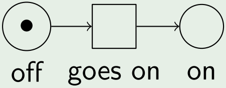

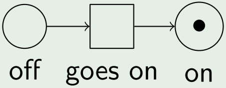

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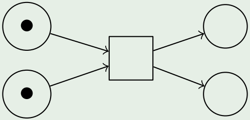
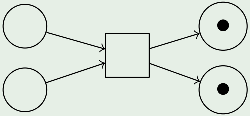
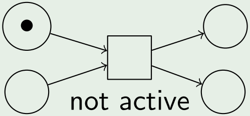
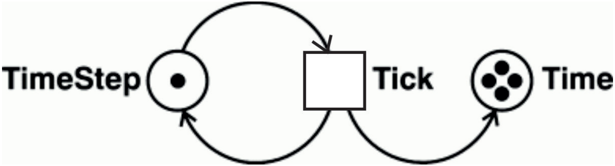


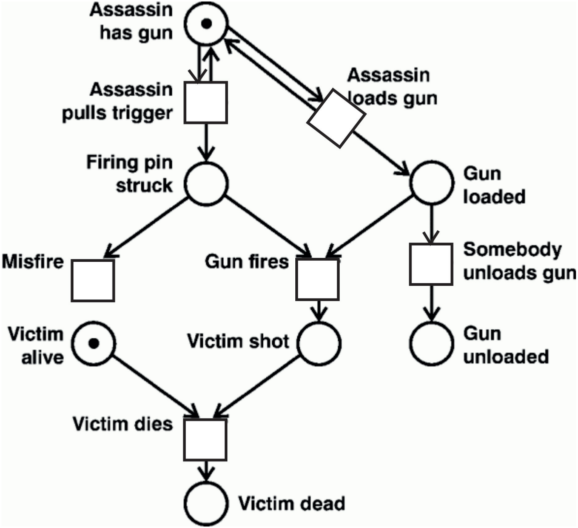




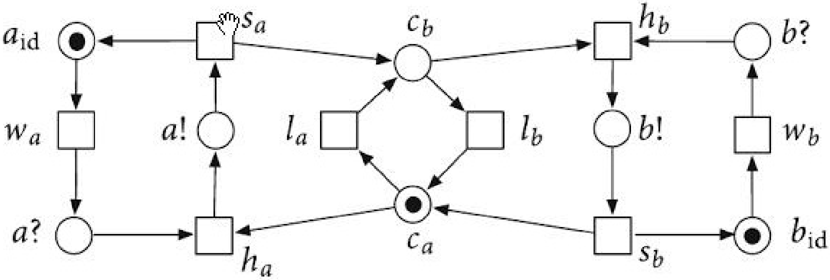
Clock at time $t = 4$:



Real world examples



Assigning a printer to two users:



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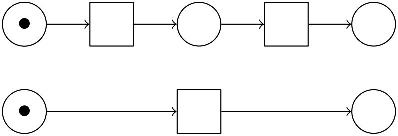

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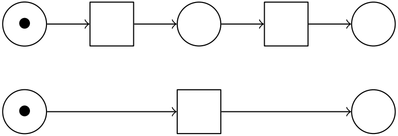

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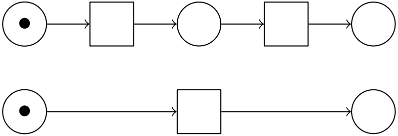

    - Reversible, can compute backwards (a key idea in physics), can recompute in case of a loss of a result.
    - Can add resources to running computations without any synchronization.

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A **Petri net** is a triple (P, T, F) with finite, disjoint sets **places** P and **transitions** T. The **flow function** $F : (P \times T) \cup (T \times P) \to \mathbb{N}$ describes interaction of places and transitions. We say that p is a **predecessor** of t if $F(p, t) > 0$, and that p is a **sucessor** of t if $F(t, p) > 0$.

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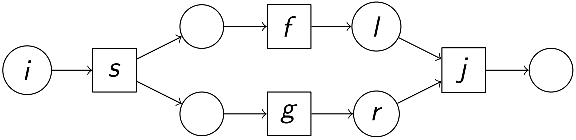
A transition enabled by M can **fire** leading to the new marking
$M'(p) = M(p) - F(p, t) + F(t, p)$.

To fire t consumes $F(p, t)$ tokens from each predecessor p of t and produces $F(q, t)$ tokens on each sucessor q of t.
To execute a Petri net we randomly fire enabled transitions.

- Task parallelism:
  Transitions f and g can fire in parallel:



Parallel computations

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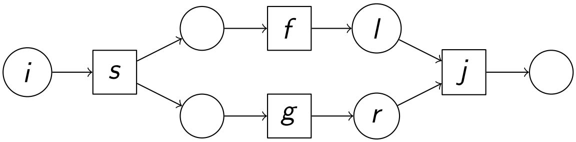
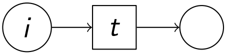


- Data parallelism:
  If i holds multiple tokens, t can fire in parallel:



Note:

- Real world transitions take time.
- Tokens can be complex data structures.

Integration by Lukas Ristau:

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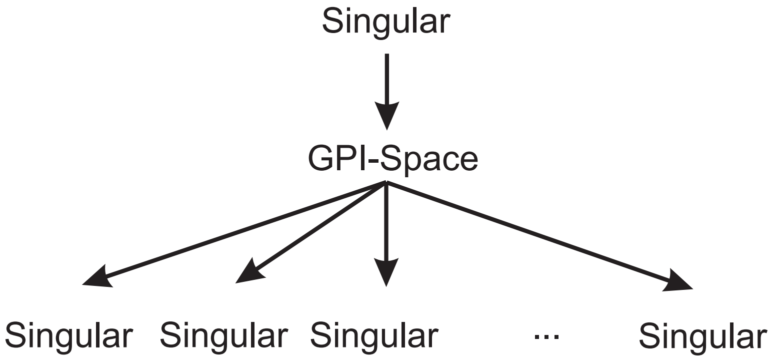
- SINGULAR calls GPI-Space.
- GPI-Space uses LIBSINGULAR on the workers.



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- Once X is completely covered with charts $U_i \subset W$ s.t. $X \cap U_i$ is a smooth complete intersection, return **true**.

Application to Algorithms in Algebraic Geometry

- Key concept in algebraic geometry:
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- Key concept in algebraic geometry:
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- Use this natural parallel structure algorithmically?
- Computational basis of most algorithms is Buchberger's Algorithm.
- Buchberger's Algorithm has doubly exponential worst case complexity [Mayr, Meyer 1982], much faster in many practical examples of interest $\rightarrow$ unpredictable for parallelization (can parallelize *individual* computations via modular and linear algebra methods).
- $\rightarrow$ Single chart may dominate the run-time.
- Solution: Model algorithm in a parallel way s.t. it automatically finds a good cover.

Campedelli surface of codim 5 with algebraic fundamental group $\mathbb{Z}/6$:

Timings for Campedelli surface

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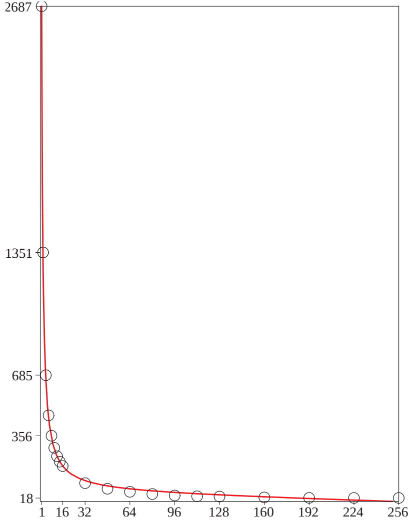
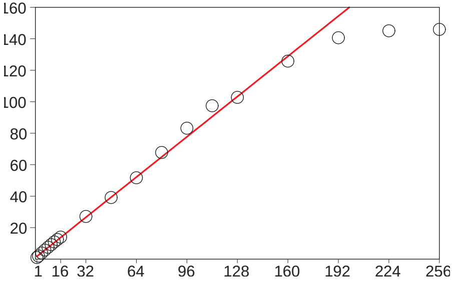


sec/cores



1/cores

Jacobian criterion takes 461sec.

Timings for Godeaux surface of codim 11

General type surface by Frank-Olaf Schreyer, Isabel Stenger.

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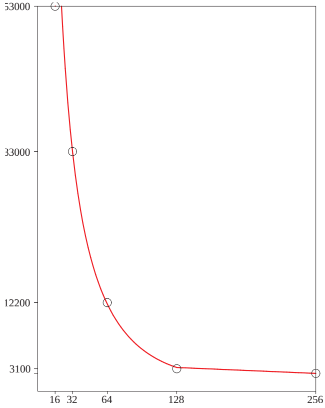

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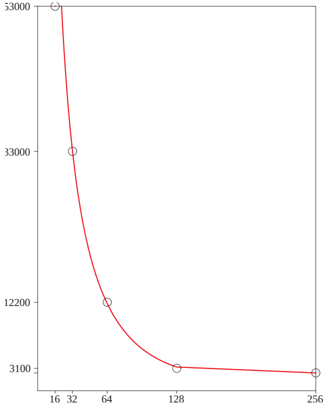

Jacobian criterion does not finish.

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- Do you have more ideas?

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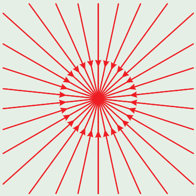

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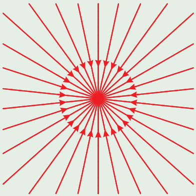

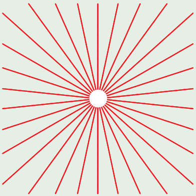

$$U//\mathbb{C}^* = \{pt\} \qquad\qquad U//\mathbb{C}^* = \mathbb{P}^1$$

GIT-Fan

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For $\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2, \quad t \cdot (x, y) = (tx, ty), \quad Q = (1, 1), \quad \mathfrak{a} = 0$

$$U_1 = \mathbb{C}^2$$
$$U_2 = \mathbb{C}^2 \backslash \{0\}$$

$\Lambda(\mathfrak{a}, Q) =$ 

Symmetries

Example

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$(\mathbb{C}^*)^2$-action on $V(\mathfrak{a})$ given by

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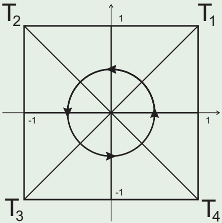
$$G = D_4 = \langle (1,2)(3,4), (1,2,3,4) \rangle \subset S_4$$



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- Traverse fan by passing through codim 1 faces to neighbours.
- Hash GIT-cones via the binary vector encoding which orbit cones occur in the corresponding intersection. Hash interacts well with symmetry group action.
- Compute in each orbit only a single representative.

Cox ring of the moduli space of stable genus zero curves with 6 marked points $\overline{M}_{0,6}$ is $\mathbb{Z}^{16}$-graded, has 40 generators (Castravet, 2009),

GIT-fans of torus actions on affine varieties

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Example

The GIT-fan decomposition of the moving cone $\mathrm{Mov}(\overline{M}_{0,6})$ classifies all small modifications (rational maps which are isomorphisms on open subsets which have a complement of codimension ≥ 2).
The moving cone $\mathrm{Mov}(\overline{M}_{0,6})$ has

$$176\,512\,225$$

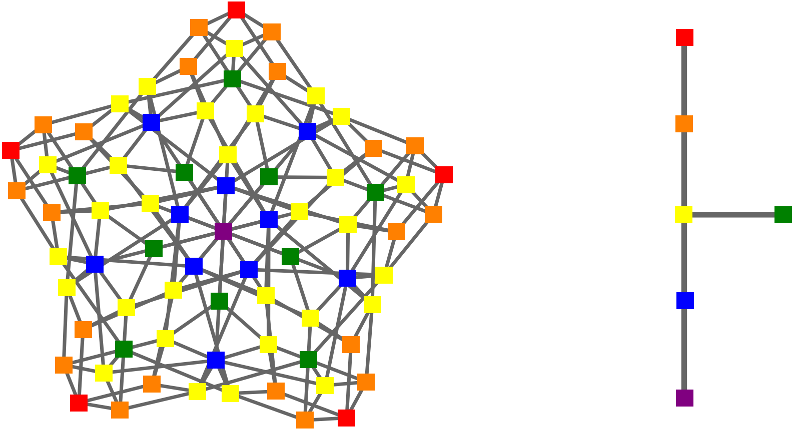
GIT-cones of maximal dimension 16, which decompose into

$$249\,605$$

orbits under the S_6-action. The cone with orbit length one is the semiample cone (dual of Mori cone).

Smaller example: GIT-fan for $\mathbb{G}(2,5)$

Adjacency graph of the maximal-dimensional GIT-cones and their orbits:

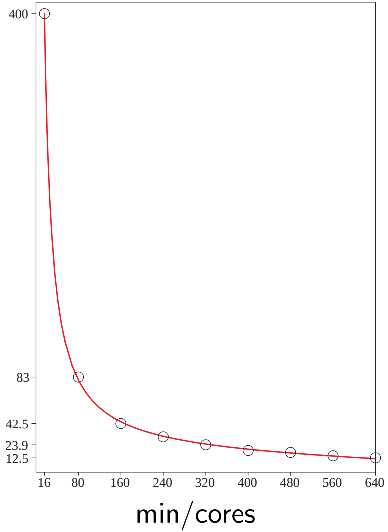


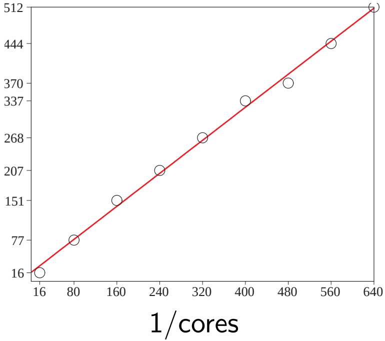
Timings and Scaling in GPI-Space for $\overline{M}_{0,6}$

Using the SINGULAR task model with 1 core 16 days, 16 cores 1 day.

Timings and Scaling in GPI-Space for $\overline{M}_{0,6}$

Using the SINGULAR task model with 1 core 16 days, 16 cores 1 day. Symmetric GIT-fan algorithm implemented by Christian Reinbold:





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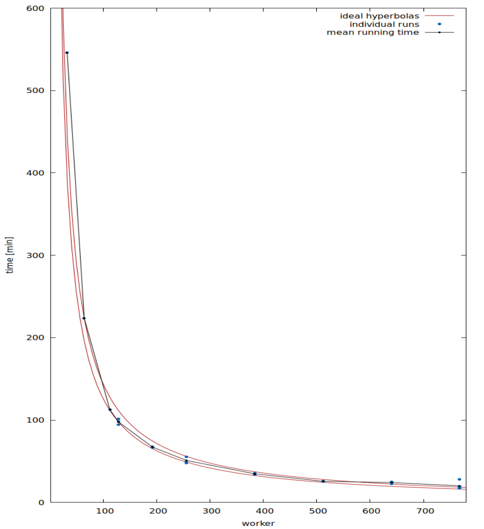
Example (Tropicalization of $\mathbb{G}(3,8)$)



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