Computing GIT-Fans with Symmetry

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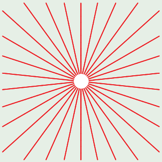

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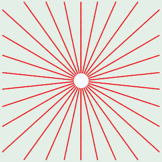

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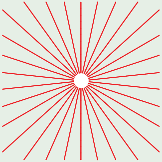

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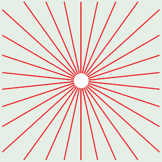

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This setting plays an important role in constructing Mori Dream Spaces.
Some moduli spaces are of this type.

Outline

- Good quotients
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- Fast monomial containment test

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- Applications

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   Hence pass to open subset $U \subset X$.

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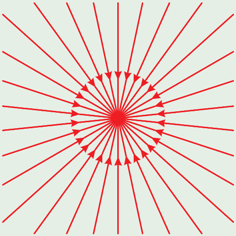

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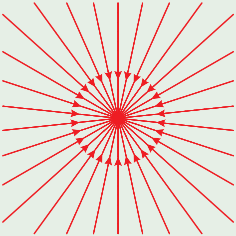

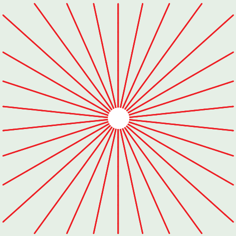

$$U /\!/ \mathbb{C}^* = \{pt\} \qquad\qquad U /\!/ \mathbb{C}^* = \mathbb{P}^1$$

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$\mathbb{C}^* \times \mathbb{C}^2 \to \mathbb{C}^2, \quad t \cdot (x, y) = (tx, ty)$ is encoded by $Q = (1, 1) \in \mathbb{Z}^{1 \times 2}$.

GIT-Fans

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Definition

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$$\omega(x) = \operatorname{cone}\{w \in \operatorname{im} Q \mid \exists f \in A_w \text{ with } f(x) \neq 0\}$$

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$w \sim w' \Leftrightarrow X^{ss}(w) = X^{ss}(w')$ *is equivalence relation.*

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$$X^{ss}(0) = \mathbb{C}^2$$
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Decomposition into torus orbits

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$$O(\gamma) = (\mathbb{C}^*)^r \cdot \sum_{e_i \in \gamma} e_i = \{(z_1, \ldots, z_r) \in \mathbb{C}^r \mid z_i \neq 0 \Leftrightarrow e_i \in \gamma\}$$

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$$\Omega = \{Q(\gamma) \mid \gamma \text{ an } \mathfrak{a}\text{-face}\}$$

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3. Determine GIT-fan:

$$\Lambda(\mathfrak{a}, Q) = \{\lambda_\Omega(w) \mid w \in \Gamma\} \quad \text{where} \quad \lambda_\Omega(w) = \bigcap_{w \in \eta \in \Omega} \eta$$

GIT-Algorithm

Algorithm

Input: *Ideal* $\mathfrak{a} \subset \mathbb{C}[T_1, \ldots, T_r]$ *and matrix* $Q \in \mathbb{Z}^{k \times r}$ *of full rank such that* $\mathfrak{a}$ *is homogeneous w.r.t. multigrading by* Q.

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3: *Choose a vector* $w_0 \in Q(\mathbb{Q}^r_{\geq 0})$ *such that* $\dim(\lambda_\Omega(w_0)) = k$.

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Input: *Ideal* $\mathfrak{a} \subset \mathbb{C}[T_1, \dots, T_r]$ *and matrix* $Q \in \mathbb{Z}^{k \times r}$ *of full rank such that* $\mathfrak{a}$ *is homogeneous w.r.t. multigrading by* Q.

Output: *The set of maximal cones of* $\Lambda(\mathfrak{a}, Q)$.

1: $\mathcal{A} := \{\gamma \prec \mathbb{Q}^r_{\geq 0} \mid \gamma \text{ is an } \mathfrak{a}\text{-face}\}$

2: $\Omega := \{Q(\gamma) \mid \gamma \in \mathcal{A}\}$

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return $\mathcal{C}$

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Example (`polymake.so`)

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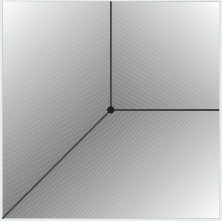
> fan F = normalFan(P); F;
  RAYS:
  -1 -1 #0
   0  1 #1
   1  0 #2
  MAXIMAL_CONES:
  {0 1} #Dimension 2
  {0 2}
  {1 2}
```



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Fast Monomial Containment Test

Generalization of (Sturmfels, 1996), where degree reverse lex (dp) is used:

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Proposition

Let $>$ be a monomial ordering on $R = K[Y_1, \ldots, Y_n]$ and $\mathcal{G}$ a Gröbner basis of I. Suppose that for all $f \in \mathcal{G}$

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Then

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To compute $I : (Y_1 \cdot \ldots \cdot Y_n)^\infty$, replace any remainder $r \neq 0$ in Buchberger's algorithm by

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$\{1,\ldots,40\}\setminus J$	$40-\lvert J\rvert$	$\mathfrak{a}$-face	divgbsat	gbsat	sat	rabinowitsch
$\{3,4,5,7,\ldots,15\}$	28	no	1	761	517	342
$\{9,11,12,13,15\}$	35	no	1	57200	$*$	$*$
$\{11,12,13,15\}$	36	no	1	44100	$*$	$*$
$\{9,11,14,15\}$	36	yes	64	121000	$*$	$*$
$\{9,11,15\}$	37	yes	1170	114000	$*$	$*$
$\{9,11,13\}$	37	no	1	31400	$*$	$*$

(in seconds, * did not finish in $>$ 2 days)

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Definition

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Perfect hash function for cones with compatible group action

Representation of GIT-Cones

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such that

$$g \cdot h_\Omega(\lambda) = h_\Omega(g \cdot \lambda).$$

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return $\mathcal{C}$

An Example with D_4-Symmetry

Example

$$\mathfrak{a} = \langle T_1 T_3 - T_2 T_4 \rangle \subset \mathbb{K}[T_1, \ldots, T_4] \qquad \deg(T_j) = q_j$$

$$Q = (q_1, \ldots, q_4) = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

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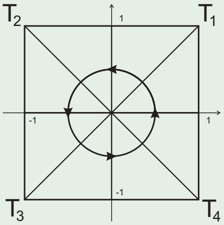

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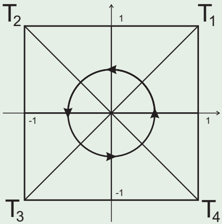

$$A_{(1,2)(3,4)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_{(1,2,3,4)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

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$\gamma_0 = \mathrm{cone}(0)$	1	0	`true`
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$\gamma_3 = \mathrm{cone}(e_1, e_2, e_3)$	4	$\langle T_1 T_3 \rangle$	`false`
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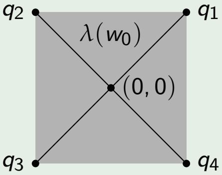

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Example: GIT-fan for $\mathbb{G}(2,5)$

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Cox ring of $\overline{M}_{0,5}$ is isomorphic to $\mathbb{Z}^5$-graded coordinate ring $R = \mathbb{K}[T_1, \ldots, T_{10}]/\mathfrak{a}$ of affine cone over $\mathbb{G}(2,5)$.

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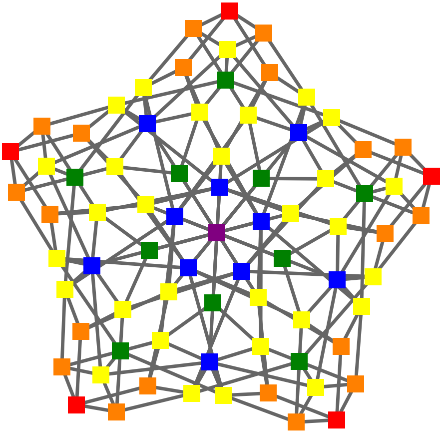
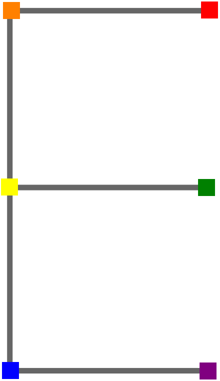
$$|\Lambda(5)| = 76 = 1 + 10 + 30 + 10 + 20 + 5$$

Adjacency graph of the maximal-dimensional GIT-cones and their orbits:





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Moving cone Mov($\overline{M}_{0,6}$) classifies all small modifications (rational maps which are isomorphisms on open subsets which have a complement of codimension ≥ 2).

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cardinality	1	6	10	15	20	30	45	60
no. of orbits	1	1	1	4	1	1	10	27

cardinality	72	90	120	180	240	360	720
no. of orbits	4	46	32	488	4	7934	241051

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