Fundamental and Advanced Algorithms in Singular

Janko Boehm, Yue Ren

University of Kaiserslautern

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Singular

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- Special emphasis on algebraic geometry, commutative and non-commutative algebra, singularity theory, packages for convex and tropical geometry.

Outline

- Warm-up, Gröbner bases
- Normalization, Adjoint Curves, Classification of Singularities
- Parallel Computations
- Resolution of Singularities
- Modular Methods
- Massively Parallel Computations

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- Primary Decomposition
- Standard Bases and Associated Graded Ring
- Convex Geometry
- Computation of Tropical Varieties

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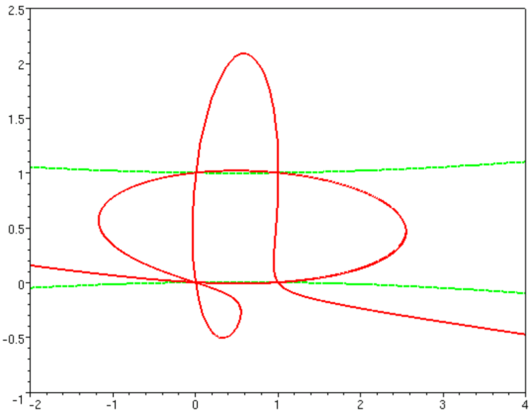
- Computing the GIT-Fan
- Feynman Integrals and Tropical Mirror Symmetry

Parametrizing Rational Curves

Example



Example

We consider the degree-5 curve with equation

$$x^5 + 10x^4y + 20x^3y^2 + 130x^2y^3 - 20xy^4 + 20y^5 - 2x^4z - 40x^3yz - 150x^2y^2z - 90xy^3z - 40y^4z + x^3z^2 + 30x^2yz^2 + 110xy^2z^2 + 20y^3z^2 = 0.$$

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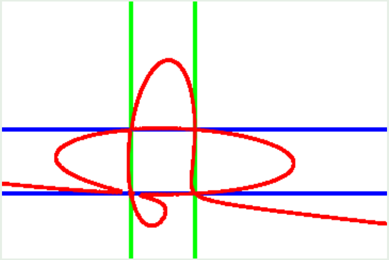

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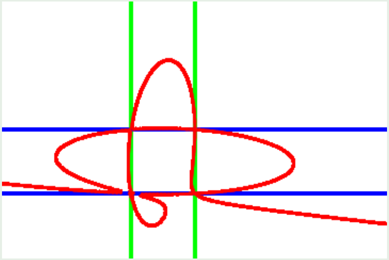

Genus Formula. $p_g(C) = p_a(C) - \delta(C) = p_a(C) - \sum_{P \in \mathrm{Sing}(\Gamma)} \delta_P(C)$

Parametrizing Rational Curves

Example

```
> ring R = 0, (x,y,z), dp;
> poly f = x5+10x4y+20x3y2+130x2y3-20xy4+20y5-2x4z-40x3yz-150x2y2z
           -90xy3z-40y4z+x3z2+30x2yz2+110xy2z2+20y3z2;
> LIB "paraplanecurves.lib";
> genus(f);
  0
> paraPlaneCurve(f);
```

Example

```
> ideal AI = adjointIdeal(f); // requires normalization, integral bases
> AI;
  _[1]=y3-y2z
  _[2]=xy2-xyz
  _[3]=x2y-xyz
  _[4]=x3-x2z
> def Rn = mapToRatNormCurve(f,AI);
> setring(Rn);
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The Parametrization Algorithm

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> RNC;
RNC[1]=y(2)*y(3)-y(1)*y(4)
RNC[2]=20*y(1)*y(2)-20*y(2)^2+130*y(1)*y(4)
       +20*y(2)*y(4)+10*y(3)*y(4)+y(4)^2
RNC[3]=20*y(1)^2-20*y(1)*y(2)+130*y(1)*y(3)
       +10*y(3)^2+20*y(1)*y(4)+y(3)*y(4)
```

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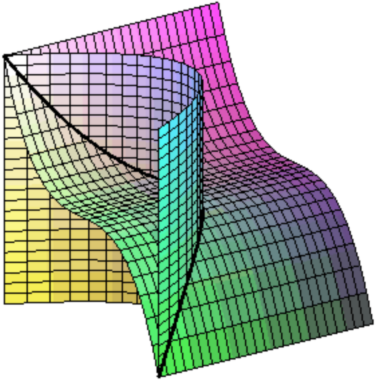

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Example

```
> LIB "sing.lib";
> radical(slocus(RNC));
  _[1]=y(4)
  _[2]=y(3)
  _[2]=y(2)
  _[1]=y(1)
> rncAntiCanonicalMap(RNC);
  _[1]=2*y(2)+13*y(4)
  _[2]=y(4)
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Remark

May require quadratic field extension in even-degree case.

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Divide $x^2 - y^2$ durch $x^2 + y$ und $xy + x$ with respect to lexicographic ordering.

$$\begin{array}{rl} x^2 - y^2 = & 1 \cdot (x^2 + y) + (-y^2 - y) \\ x^2 + y & \\ \hline -y^2 - y & \end{array}$$

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so remainder $\neq 0$, but

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$$f \in I \Longleftrightarrow NF(f, G) = 0$$

The Main Computational Tool: Gröbner Bases

Example

Gröbner Bases can be used to:

- eliminate variables ($\rightarrow$ birational geometry),
- ideal intersections,
- compute ideal quotients

  $$(I : J) = \{a \in R \mid aJ \subset I\}$$

  for ideals $I, J \subset R$,
- saturations,
- syzygies ($\rightarrow$ homological algebra).

Greuel, G.-M., Pfister, G.: *A Singular Introduction to Commutative Algebra.* Springer.

Normalization

Setup: $A = K[X]/I$ domain.

Definition

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Curve $I = \langle x^3 + x^2 - y^2 \rangle \subset K[x, y]$

$$\begin{array}{ccccc} A = K[x,y]/I & \cong & K[t^2-1, t^3-t] & \subset & K[t] \cong \overline{A} \\ \overline{x} & \mapsto & t^2 - 1 & & \\ \overline{y} & \mapsto & t^3 - t & & \end{array}$$

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As an A-module $\overline{A} = \left\langle 1, \frac{\overline{y}}{\overline{x}} \right\rangle$.

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we get a chain of extensions of reduced Noetherian rings

$$A = A_0 \subset \cdots \subset A_i \subset \cdots \subset A_m = A_{m+1}.$$

Terminates since A is Noetherian.

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Lemma

$N(A_i) \subset V(\sqrt{JA_i})$

Theorem (BDLSS, 2011)

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and

$$\overline{A} = B_1 + \ldots + B_r.$$

We call B_i *the* **minimal local contribution** *to* $\overline{A}$ *at* P_i.

Adjoint ideals

Setup: $\Gamma \subset \mathbb{P}^r$ integral, non-degenerate projective curve, $\pi : \overline{\Gamma} \to \Gamma$ normalization map, $I(\Gamma) \subsetneq I \subset k[x_0, ..., x_r]$ saturated homogeneous ideal.

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gives for $m \gg 0$ linear maps

$$0 \to I_m / I(\Gamma)_m \stackrel{\overline{\varrho_m}}{\to} H^0\left(\overline{\Gamma}, \mathcal{O}_{\overline{\Gamma}}\left(mH - \Delta(I)\right)\right) \to H^0\left(\Gamma, \mathcal{F}\right) \to 0$$

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I is an **adjoint ideal** of Γ if $\overline{\varrho_m}$ surjective for $m \gg 0$.

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$$h^0\left(\Gamma, \mathcal{F}\right) = \sum\nolimits_{P \in \mathrm{Sing}(\Gamma)} \ell(I_P \overline{\mathcal{O}_{\Gamma,P}} / I_P) \qquad \Longrightarrow$$

Theorem

$$I \text{ adjoint} \iff I_P \overline{\mathcal{O}_{\Gamma,P}} = I_P \text{ for all } P \in \mathrm{Sing}(\Gamma).$$

Conductor is largest ideal with this property.

Definition

Gorenstein adjoint ideal is the unique largest homogeneous ideal $\mathfrak{G} \subset K[x_0, \ldots, x_r]$ with

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Example

Brill-Noether-Algorithm for computing Riemann-Roch spaces.

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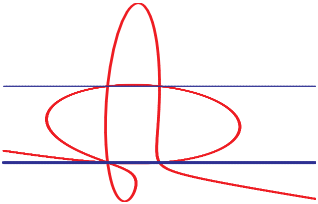

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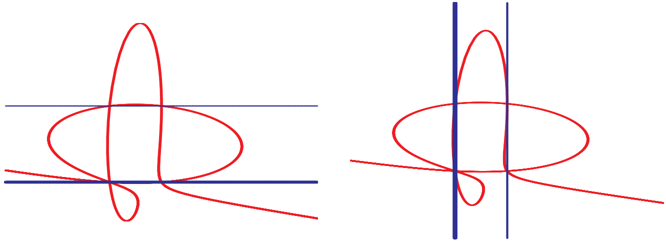

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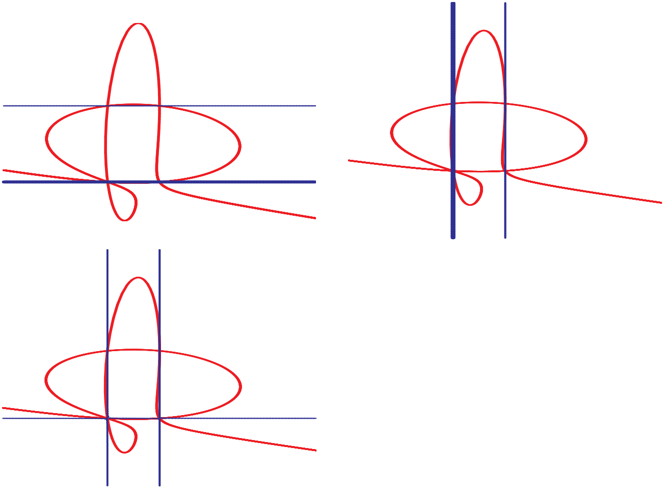

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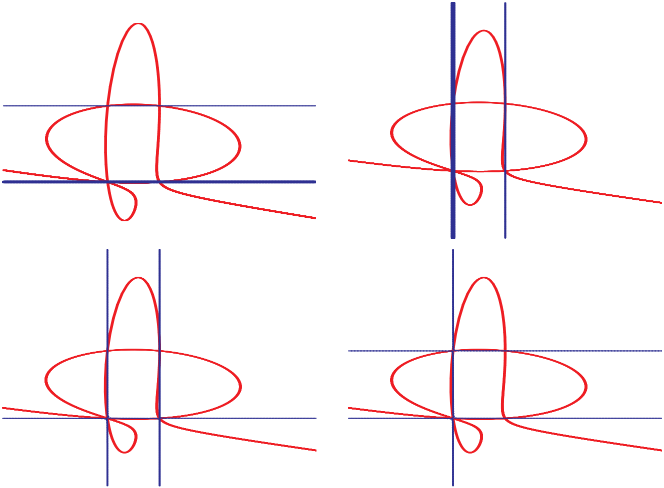

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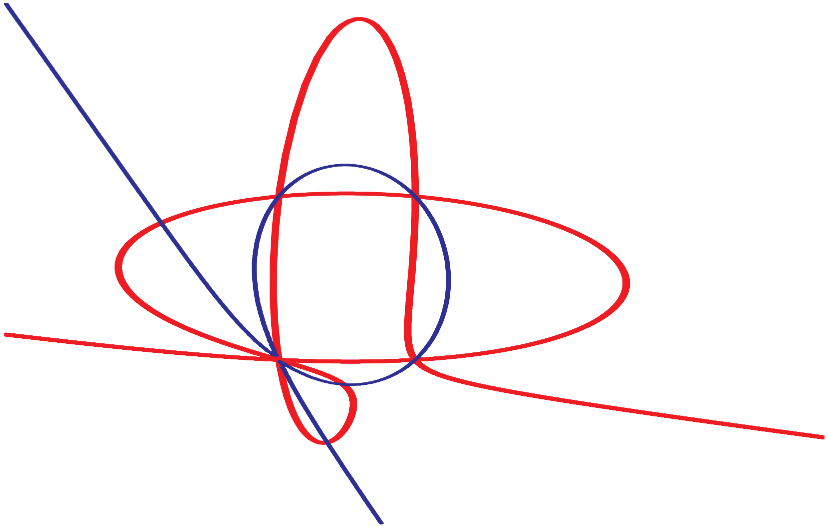

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Lemma (BDLP, 2015)

$$\mathfrak{G} = \bigcap_{P \in \operatorname{Sing}\Gamma} \mathfrak{G}(P)$$

The $\mathfrak{G}(P)$ can be computed in parallel via normalization.

Local-to-global algorithm

Definition

The **local adjoint ideal** of Γ at $P \in \operatorname{Sing}\Gamma$ is the largest homogeneous ideal $\mathfrak{G}(P) \subset k[x_0, \ldots, x_r]$ with

$$\mathfrak{G}(P)_P = \mathcal{C}_{\mathcal{O}_{\Gamma,P}}$$

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Algorithm (BDLP, 2015)

If $\frac{1}{d}U$ is the minimal local contribution at P then

$$\mathfrak{G}(P) = (d : U)^h$$

If $\Gamma \subset \mathbb{P}^2$ has a singularity of type A_n at $P = (0:0:1)$, then given by

$$f = T^2 + W^{n+1} \quad \text{with} \quad T, W \in \mathbb{C}[[x, y]].$$

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If $P = (0,0)$ is of type A_n and $s = \left\lfloor \frac{n+1}{2} \right\rfloor$, then

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Similar results for D_n, E_n and other singularities in Arnold's list.

Example

$f = x^4 - y^2 + x^5$ with A_3 singularity. Then $\mathfrak{G}(P) = \langle x^2, y \rangle$.

Example

```
> LIB("parallel.lib","random.lib");
> ring R = 0,x(1..4),dp;
> ideal I = randomid(maxideal(3),3,100);
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Parallel Computations in SINGULAR

Example

```
> LIB("parallel.lib","random.lib");
> ring R = 0,x(1..4),dp;
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> proc sizeStd(ideal I, string monord){
      def R = basering; list RL = ringlist(R);
      RL[3][1][1] = monord; def S = ring(RL); setring(S);
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> parallelWaitAll(commands, args);
  [1] 55
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```

Parallelization

There are algorithms whose basic strategy is inherently parallel, whereas others are sequential in nature.

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Example

- Normalization is inherently sequential.
- Local-to-global algorithms for normalization and adjoint ideal are parallel, if the singular locus decomposes.
- Villamayor's constructive version of Hironaka's desingularization theorem is inherently parallel by the iterative use of blow-ups in charts.
- Modular methods can be used to turn sequential algorithms over $\mathbb{Q}$ into parallel ones.

Example: Resolution of Singularities

Theorem (Hironaka, 1964)

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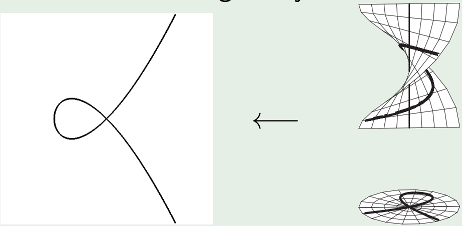
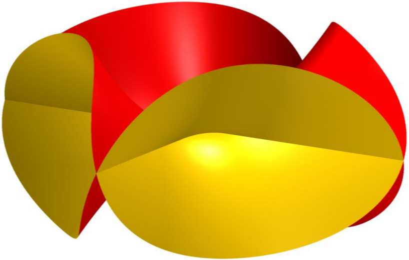
Example

Blow-up of the node resolves the singularity



by replacing it by a line of points corresponding to its tangent directions, hence separating the two branches of the curve.

Example:
$x^2 - y^2z^2 = 0$



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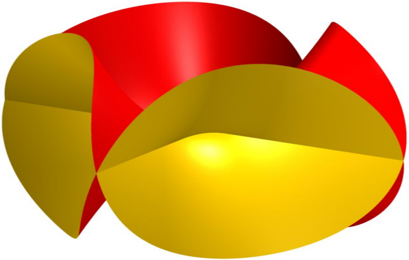

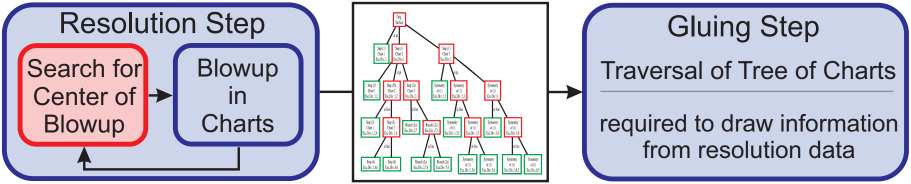

Hironaka Resolution of Singularities

Example

```
> LIB "resolve.lib";
> ring R= 0,(x,y,z),dp;
> ideal I = x2-y2z2;
> list L = resolve(I);
> def S1 = L[1][1];
> setring S1;
> showBO(BO);
  ==== Ambient Space:
  _[1]=0
  ==== Ideal of Variety:
  _[1]=y(1)^2-1
  ==== Exceptional Divisors:
  ...
```

Modular computations

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- Goal:
  General reconstruction scheme for algorithms in commutative algebra, algebraic geometry, number theory.

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Compute

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How to obtain a rational number from $\overline{22684}$?

Theorem (Kornerup, Gregory, 1983)

The **Farey map**

$$\left\{ \tfrac{a}{b} \in \mathbb{Q} \;\middle|\; \begin{array}{c} \gcd(a,b)=1 \\ \gcd(b,N)=1 \end{array} \quad |a|\,,|b| \le \sqrt{(N-1)/2} \right\} \longrightarrow \mathbb{Z}/N$$

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Indeed, in the above example

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Basic concept for modular computations

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Definition

A prime p is called **bad** if the result over $\mathbb{Q}$ does not reduce modulo p to the result over $\mathbb{Z}/p$.

Bad primes in Gröbner basis computations

For $G \subset K[X] = K[x_1, \ldots, x_n]$ and a monomial ordering $>$, let $\mathrm{LM}(G)$ be the set of lead monomials of G.

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that is, p is not bad.

Bad primes in Gröbner basis computations

Example

```
> option("redSB");
> ring R = integer,(x, y, z),lp;
> poly f = x7y5 + x2yz9 + xz11 + y3z9;
> ideal I = groebner(ideal(diff(f, x), diff(f, y), diff(f,z)));
> apply(list(I[1..size(I)]),leadcoef);
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12 4 12 12 45349632 12 1473863040 12 22674816 12 3888 12 12 12 13608 12
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1378111552786873034477731046461326O 835219122901135172410746O8876444 60
12 4 12 12 45349632 12 1473863040 12 22674816 12 3888 12 12 12 13608 12
108 54 6 2 27 3 1 4 2 2 1 216 1 2 3 1 540 12 108 27 3 1 9 3 1 1 1 1 1 7
1 5 1 1
```

and the bad primes are the prime factors

$$p = 2, 3, 5, 7, 11, 13, 257, 247072949, 3288380889935506827$$

Bad primes in Gröbner basis computations

Example

```
> option("redSB");
> ring R = integer,(x, y, z),lp;
> poly f = x7y5 + x2yz9 + xz11 + y3z9;
> ideal I = groebner(ideal(diff(f, x), diff(f, y), diff(f,z)));
> apply(list(I[1..size(I)]),leadcoef);
137811552786873034477310464613260 83521912290113517241074608876444 60
12 4 12 12 45349632 12 1473863040 12 22674816 12 3888 12 12 12 13608 12
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Note: The lead coefficients of the Gröbner basis over $\mathbb{Q}$ involve only the prime factors $2, 3, 5, 7, 13$.

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- Type 5: otherwise.

Example of type 5 bad prime

For ideal $I \subset \mathbb{Q}[X]$ and prime p define $I_p = (I \cap \mathbb{Z}[X])_p$.

Example

Consider the algorithm $I \mapsto \sqrt{I + \mathrm{Jac}(I)}$ for

$$I = \langle x^6 + y^6 + 7x^5z + x^3y^2z - 31x^4z^2 - 224x^3z^3 + 244x^2z^4 + 1632xz^5 + 576z^6 \rangle$$

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Hence

$$U(0)_5 \neq U(5)$$

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Now suppose

$$N = N' \cdot M$$

with $\gcd(N', M) = 1$.

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$$(a^2 + b^2)M < N',$$

then (by the lemma)

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and such vectors exist. Moreover, if $\gcd(a, b) = 1$ *and* (x, y) *is a shortest vector* $\neq 0$ *in* Λ, *we also have* $\gcd(x, y) | M$.

Error tolerant reconstruction via Gauss-Lagrange

Hence, if $N' \gg M$, the Gauss-Lagrange-Algorithm for finding a shortest vector $(x, y) \in \Lambda$ gives $\frac{a}{b}$ independently of t, provided $x^2 + y^2 < N$.

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Algorithm (Error tolerant reconstruction)

```
function ErrorTolerantReconstruction(r::Integer, N::Integer)
a1 = [N, 0]
a2 = [r, 1]
while dot(a1, a1) > dot(a2, a2)
q = dot(a1, a2)//dot(a2, a2)
a1, a2 = a2, a1 - Integer(round(q))*a2
end
if dot(a1, a1) < N
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else
return false
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Singular-kernel	Julia	Singular-interpreter
0.001	0.005	0.055

(in seconds, bitlength 500)

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Example

We reconstruct $\frac{13}{12}$ from

$$\overline{22684} \in \mathbb{Z}/38885$$

by determining a shortest vector in the lattice

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$$\begin{aligned}
(38885, 0) &= 2 \cdot (22684, 1) + (-6483, -2), \\
(22684, 1) &= -3 \cdot (-6483, -2) + (3235, -5), \\
(-6483, -2) &= 2 \cdot (3235, -5) + (-13, -12), \\
(3235, -5) &= -134 \cdot (-13, -12) + (1493, -1613).
\end{aligned}$$

Example

Now introduce an error in the modular results:

$$\begin{array}{ccccccccc} \mathbb{Z}/5 & \times & \mathbb{Z}/7 & \times & \mathbb{Z}/11 & \times & \mathbb{Z}/101 & \cong & \mathbb{Z}/38885 \\ (\ \overline{4} & , & \overline{4} & , & \overline{2} & , & \overline{60}\ ) & \mapsto & \overline{22684} \end{array}$$

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$$(38885, 0) = 84 \cdot (464, 1) + (-91, -84),$$
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Note that

$$(13^2 + 12^2) \cdot 7 = 2191 < 5555 = 5 \cdot 11 \cdot 101.$$

General reconstruction scheme

Setup: For ideal $I \subset \mathbb{Q}[X]$ compute ideal (or module) $U(0)$ associated to I by deterministic algorithm.

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Theorem (BDFP, 2015)

If the set of bad primes for computing $U(0)$ from I is finite, then this algorithm terminates with the correct result.

Timings in Singular for Adjoint Ideal

Plane curve f_n of degree n with $\binom{n-1}{2}$ singularities of type A_1.

Timings in SINGULAR for Adjoint Ideal

Plane curve f_n of degree n with $\binom{n-1}{2}$ singularities of type A_1.

	parallel	probablisitic	f_5		f_6		f_7	
`locNormal`			2.1		56		-	
`Maple-IB`			5.1		47		318	
`LA`			98		4400		-	
`IQ`			1.3		54		3800	
`locIQ`	■		1.3	(1)	54	(1)	3800	(1)
`ADE`	■		.18	(1)	1.2	(1)	49	(1)
`modLocIQ`			6.4	[33]	19	[53]	150	[75]
		■	6.2	[33]	18	[53]	104	[75]
	■		.36	(74)	1.6	(153)	51	(230)
	■	■	.21	(74)	0.48	(153)	5.2	(230)

[primes] (cores)

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Singular and GPI-Space

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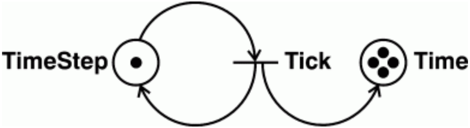
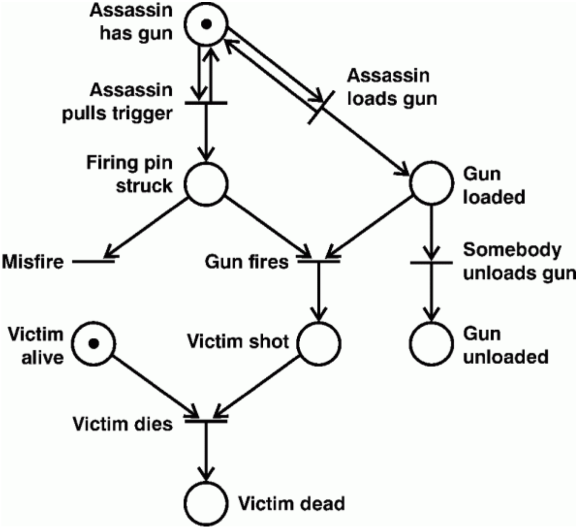
GPI-Space: A Petri net

Clock at time $t = 4$:



GPI-Space: A Petri net



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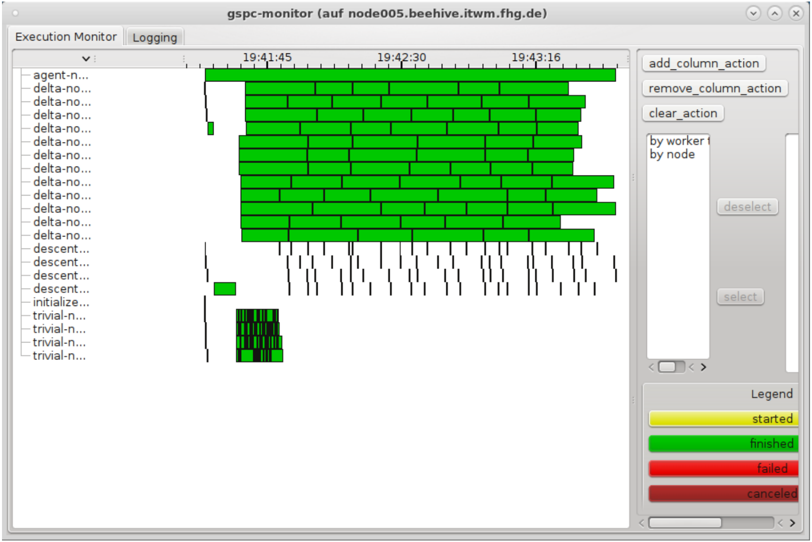
GPI-Space: Scheduler



Singular and GPI-Space

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Integration of SINGULAR in GPI-Space.
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Algorithm for determining smoothness by local descent in codimension relative to a smooth complete intersection (as in Hironaka's resolution of singularities). Descent to any desired size of minors in Jacobian criterion.

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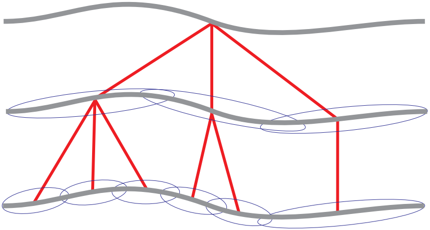

Boehm, J., Frühbis-Krüger: *A smoothness test for higher codimensions.* arXiv:1603.09241 JSC (to appear).

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   Hence pass to open subset $U \subset X$.

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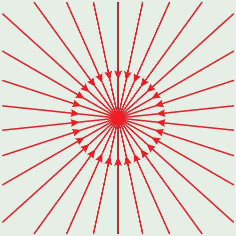

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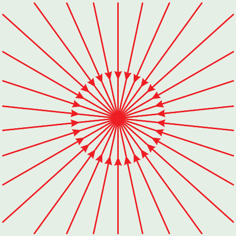
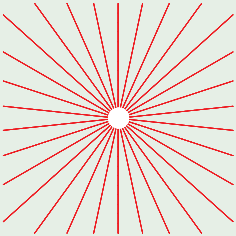
$U = \mathbb{C}^2$ | $U = \mathbb{C}^2 \setminus \{0\}$





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GIT-Fan

In general, there are many choices for these open subsets $U \subset X$ leading to different quotients. To describe this behaviour, Dolgachev and Hu introduced the **GIT-fan**, a polyhedral fan describing the variation of GIT-quotients.

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$$\Lambda(\mathfrak{a}, Q) = \{\lambda_\Omega(w) \mid w \in \Gamma\} \quad \text{where} \quad \lambda_\Omega(w) = \bigcap_{w \in \eta \in \Omega} \eta$$

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Input: *Ideal* $\mathfrak{a} \subset \mathbb{C}[T_1, \ldots, T_r]$ *and matrix* $Q \in \mathbb{Z}^{k \times r}$ *of full rank such that* $\mathfrak{a}$ *is homogeneous w.r.t. multigrading by* Q.

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$$\frac{r}{Y_1^{a_1} \cdot \ldots \cdot Y_n^{a_n}} \quad \text{where } a_j \text{ is maximal s.t. } Y_j^{a_j} \mid r.$$

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$\{1,\ldots,40\}\setminus J$	$40-\lvert J\rvert$	$\mathfrak{a}$-face	divgbsat	gbsat	sat	rabinowitsch
$\{3,4,5,7,\ldots,15\}$	28	no	1	761	517	342
$\{9,11,12,13,15\}$	35	no	1	57200	$*$	$*$
$\{11,12,13,15\}$	36	no	1	44100	$*$	$*$
$\{9,11,14,15\}$	36	yes	64	121000	$*$	$*$
$\{9,11,15\}$	37	yes	1170	114000	$*$	$*$
$\{9,11,13\}$	37	no	1	31400	$*$	$*$

(in seconds, * did not finish in >2 days)

Symmetry Groups of Torus Actions

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$$\begin{array}{ccc} e_j & \longmapsto & e_{\sigma(j)} \\ \mathbb{Q}^r & \longrightarrow & \mathbb{Q}^r \\ Q \downarrow & & \downarrow Q \\ \mathbb{Q}^k & \underset{A_\sigma}{\longrightarrow} & \mathbb{Q}^k \end{array}$$

Representation of GIT-Cones

Perfect hash function for cones with compatible group action

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$$h_\Omega\colon \Lambda(\mathfrak{a}, Q) \to \{0,1\}^\Omega, \qquad \lambda \mapsto \left[\begin{array}{l} \Omega \to \{0,1\} \\ \vartheta \mapsto \begin{cases} 1 & \lambda \subset \vartheta \\ 0 & \lambda \not\subset \vartheta \end{cases} \end{array} \right]$$

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such that

$$g \cdot h_\Omega(\lambda) = h_\Omega(g \cdot \lambda).$$

Symmetric GIT-Algorithm

Algorithm (System of representatives of the G-orbits on $\Lambda(\mathfrak{a}, Q)(k)$)

1: $\mathcal{S}$:= *system of representatives of G-orbits of* $\mathrm{faces}(\mathbb{Q}^r_{\geq 0})$

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return $\mathcal{C}$

An Example with D_4-Symmetry

Example

$$\mathfrak{a} = \langle T_1 T_3 - T_2 T_4 \rangle \subset \mathbb{K}[T_1, \ldots, T_4] \qquad \deg(T_j) = q_j$$

$$Q = (q_1, \ldots, q_4) = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

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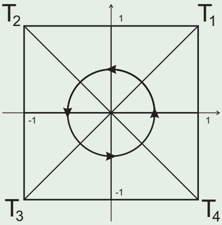

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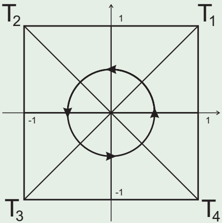

$$A_{(1,2)(3,4)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_{(1,2,3,4)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Example with D_4-symmetry

Example

γ	$\lvert G\cdot\gamma\rvert$	$\mathfrak{a}\vert_{T_i=0 \text{ for } e_i\notin\gamma}$	$\mathfrak{a}$-face
$\gamma_0 = \text{cone}(0)$	1	0	`true`
$\gamma_1 = \text{cone}(e_1)$	4	0	`true`
$\gamma_2 = \text{cone}(e_1, e_2)$	4	0	`true`
$\gamma_2' = \text{cone}(e_1, e_3)$	2	$\langle T_1 T_3\rangle$	`false`
$\gamma_3 = \text{cone}(e_1, e_2, e_3)$	4	$\langle T_1 T_3\rangle$	`false`
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$$Q(\gamma_0) = \text{cone}(0), \quad Q(\gamma_1) = \text{cone}\begin{bmatrix}1\\1\end{bmatrix}, \quad Q(\gamma_2) = \text{cone}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right), \quad Q(\gamma_4) = \mathbb{Q}^2$$

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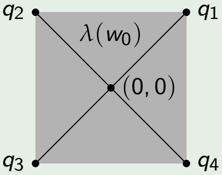
$$w_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



Mori Dream Spaces

A projective variety X over $\mathbb{C}$ is called a **Mori dream space** if its Cox ring $R(X) = \sum_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$ is finitely generated.

Example

- Fano varieties.
- Projective toric varieties ($\Leftrightarrow R(X)$ polynomial ring).

Like toric varieties, admit construction as GIT-quotient (Hu, Keel, 2000):

$$X = \hat{X} /\!/ G$$

where

$$\begin{array}{ccc} \hat{X} & \subset & \overline{X} := \operatorname{Spec} R(X) \text{ open invariant} \\ & & G := \operatorname{Spec} \mathbb{C}[\mathrm{Cl}(X)] \end{array}$$

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Remark

The GIT-fan yields the Mori chamber decomposition, which describes all birational modifications (analogous to the GKZ-fan of a toric varietiy).

Moduli Spaces of Stable Maps

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- $\overline{M}_{0,n}$ for $n \geq 10$ is not a Mori dream space:
  - Castravet, Tevelev, 2013, for $n \geq 134$.
  - Gonzáles, Karu, 2016, for $n \geq 13$.
  - Hausen, Keicher, Laface, 2016, for $n \geq 10$.

Example: GIT-fan for $\mathbb{G}(2,5)$

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Cox ring of $\overline{M}_{0,5}$ is isomorphic to $\mathbb{Z}^5$-graded coordinate ring $R = \mathbb{K}[T_1, \ldots, T_{10}]/\mathfrak{a}$ of affine cone over $\mathbb{G}(2,5)$.

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	number	number of orbits
monomial containment tests	$2^{10} = 1024$	34
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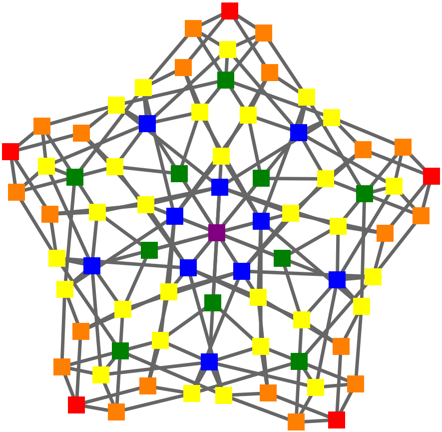
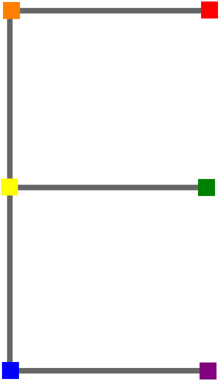
$$|\Lambda(5)| = 76 = 1 + 10 + 30 + 10 + 20 + 5$$

Adjacency graph of the maximal-dimensional GIT-cones and their orbits:





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The moving cone Mov($\overline{M}_{0,6}$) classifies all small modifications (rational maps which are isomorphisms on open subsets which have a complement of codimension ≥ 2).

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cardinality	1	6	10	15	20	30	45	60
no. of orbits	1	1	1	4	1	1	10	27

cardinality	72	90	120	180	240	360	720
no. of orbits	4	46	32	488	4	7934	241051

The cone with orbit length one is the semiample cone (dual of Mori cone).

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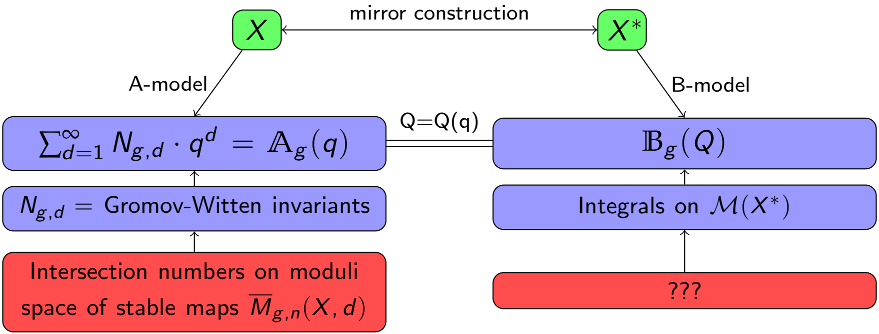

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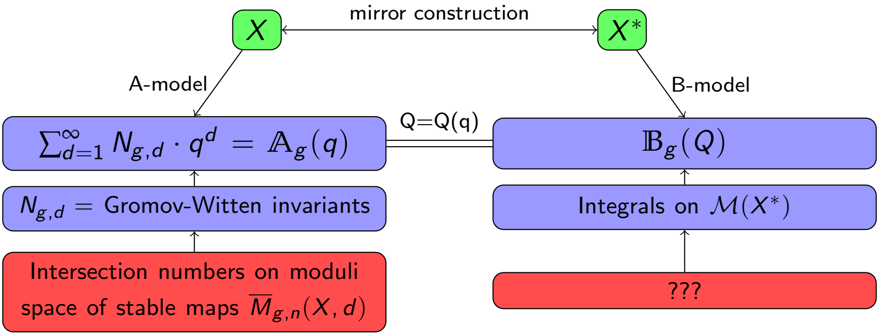

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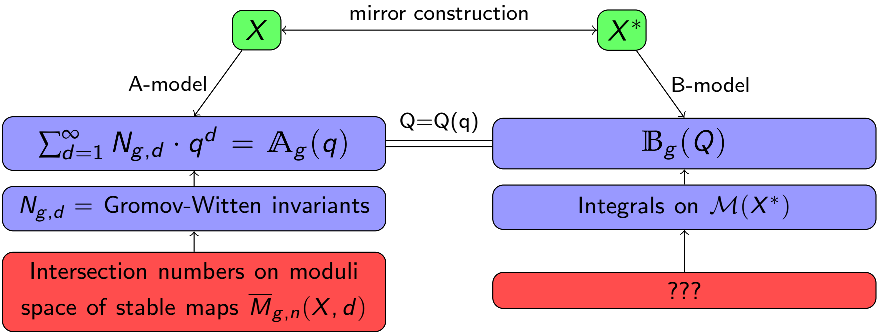

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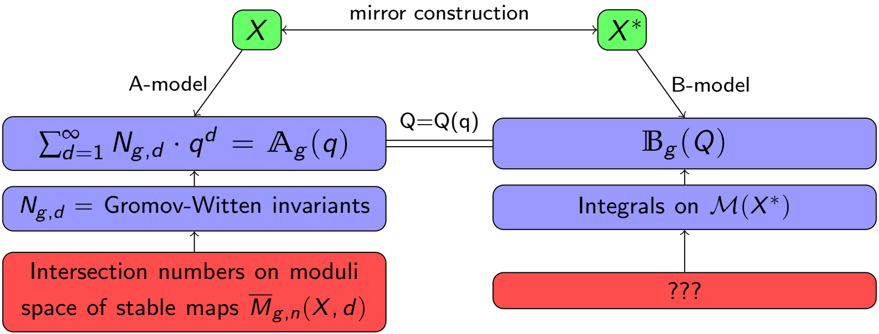

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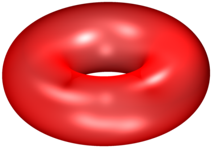
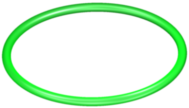
$N_{d,0} = 0$, so have to look at $g \geq 1$ invariants!

How to understand *all* $N_{g,d}$? Pass to **tropical geometry**:

$$E \quad \mapsto \quad \mathrm{trop}(E)$$





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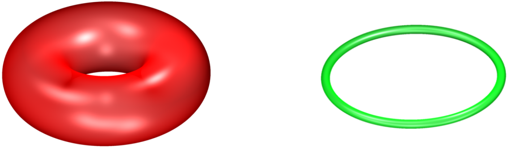

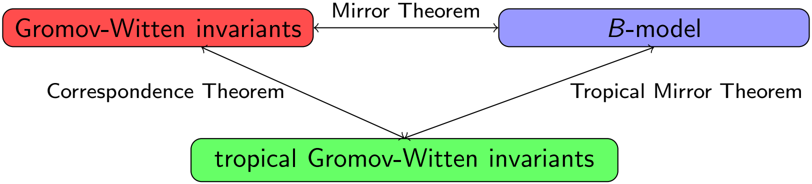

Tropical point of view

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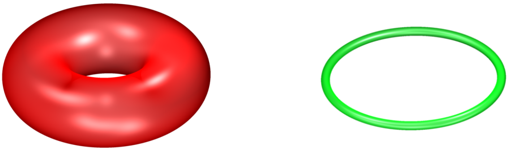

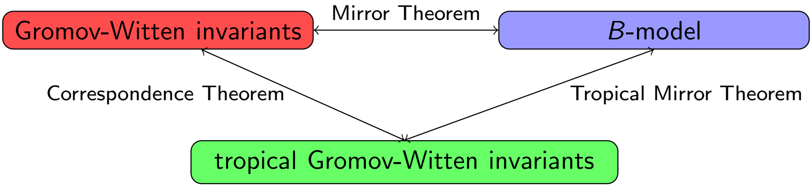

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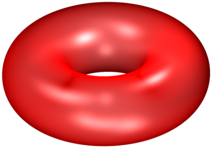

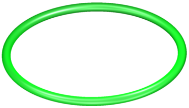

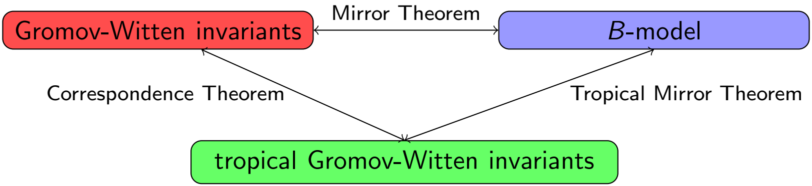

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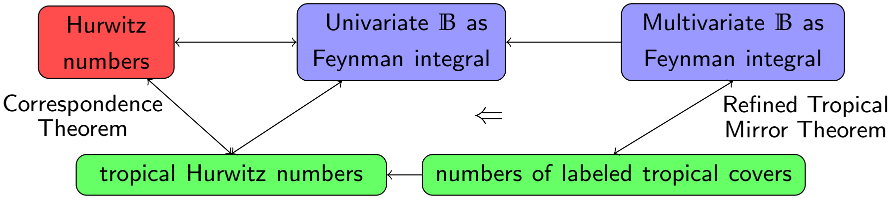
- tropical mirror theorem (Gross '10)
- partial correspondence theorem (Markwig-Rau '09, Mikhalkin '05)

Tropical Mirror Symmetry



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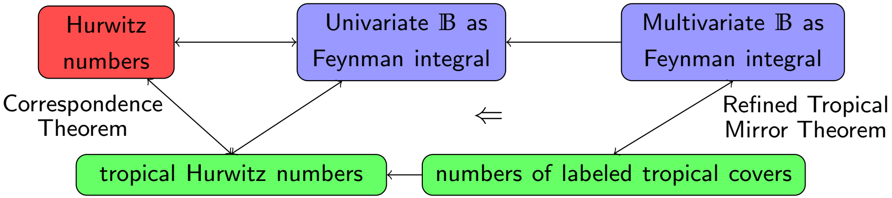
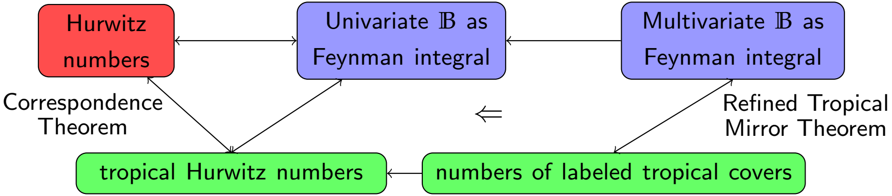


- Correspondence theorem for all g and d.



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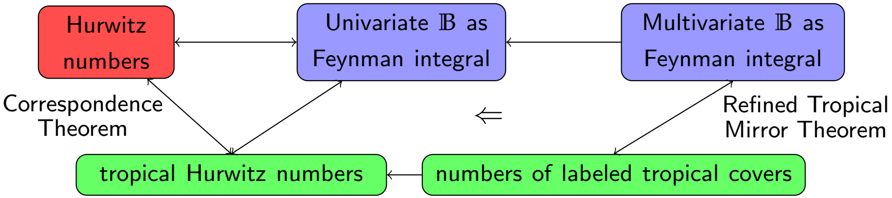

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- refined tropical mirror theorem for each trivalent connected graph of genus g and branch type.
- Computation of refined Feynman integrals.

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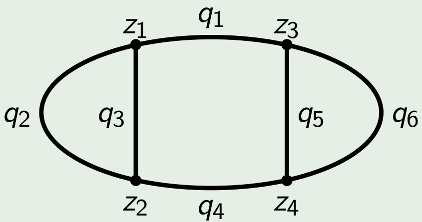

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For ordering $\Omega \in S_{2g-2}$ of integration paths on E

$$I_{\Gamma,\Omega} = \int_{\gamma_{2g-2}} \ldots \int_{\gamma_1} \left(\prod_{e\in \text{edges}(\Gamma)} P(z_e^+ - z_e^-, q) \right) dz_{\Omega(1)} \ldots dz_{\Omega(2g-2)}$$

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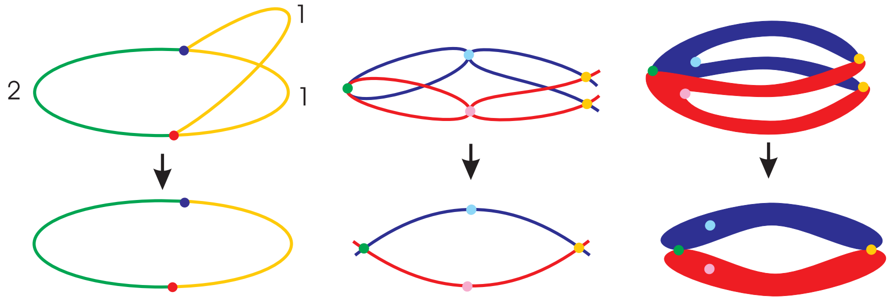

Tropical Hurwitz numbers – Example

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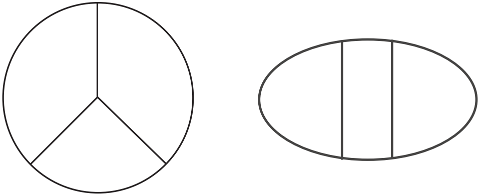
Two trivalent, connected combinatorial types (non-metric graphs)



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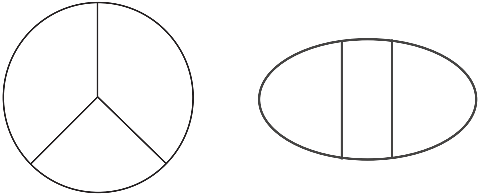

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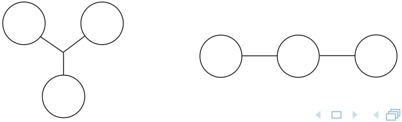

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$$N_{3,3}^{trop} =$$

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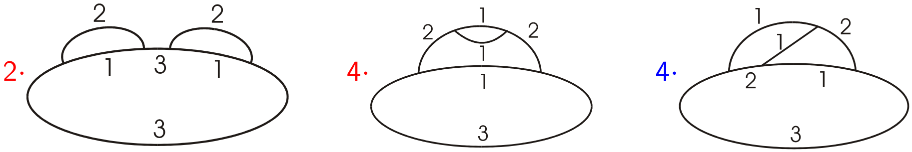

$\text{mult}(\pi) = 2^2 \cdot 3^2 = 36$

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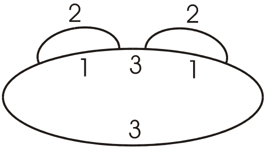
$2\cdot$



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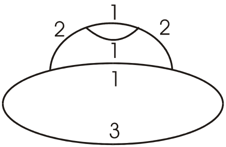

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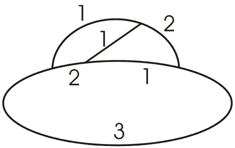

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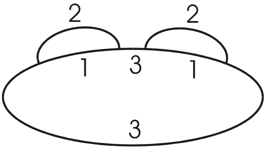

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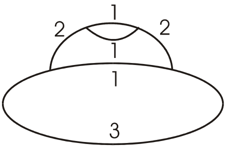

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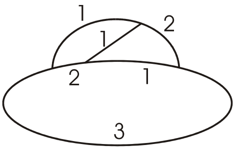

$\text{mult}(\pi) = 2^2 \cdot 3 = 12$

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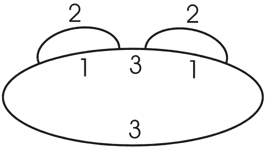

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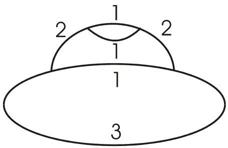

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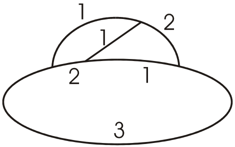

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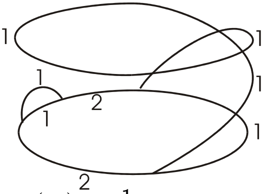
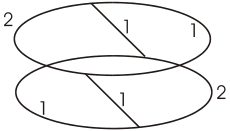


$\text{mult}(\pi) = \frac{1}{2} \cdot 2 \cdot 2 = 2$

2·



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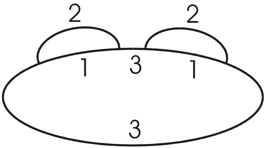

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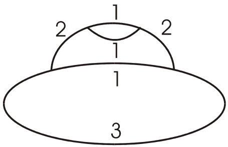

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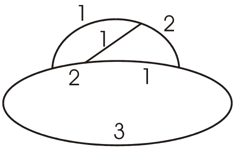

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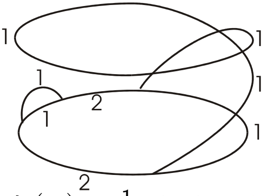

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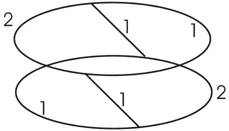
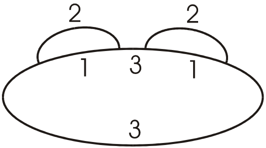


$\mathrm{mult}(\pi) = 2^2 = 4$

$$N_{3,3}^{trop} = 112 + 48 = 160$$

$2\cdot$



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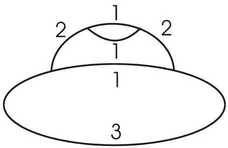

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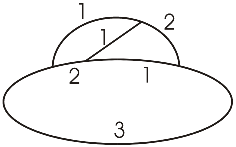

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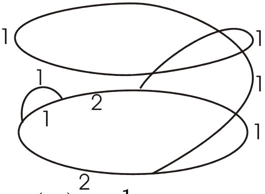

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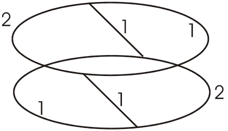

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Refined Feynman integrals

Definition (Refined Feynman integrals)

$$I_{\Gamma,\Omega}(q_1, ..., q_{3g-3}) = \int_{\gamma_{2g-2}} ... \int_{\gamma_1} \left(\prod_{k=1}^{3g-3} P(z_k^+ - z_k^-, q_k) \right) dz_{\Omega(1)} ... dz_{\Omega(2g-2)}$$

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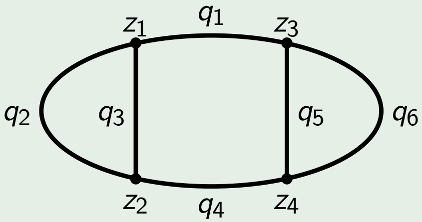
Example

For



we have to integrate

$$P(z_1 - z_2, q_1) \cdot P(z_1 - z_2, q_2) \cdot P(z_1 - z_3, q_3) \cdot P(z_2 - z_4, q_4) \cdot P(z_3 - z_4, q_5) \cdot P(z_3 - z_4, q_6)$$

Theorem (Multivariate tropical mirror theorem, BBBM '13)

$$\sum_{\underline{a}} N^{trop}_{\underline{a},\Gamma,\Omega}\; q^{2\underline{a}} = I_{\Gamma,\Omega}(q_1, \ldots, q_{3g-3})$$

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Together with the correspondence theorem this proves:

Corollary (Mirror symmetry for elliptic curves)

For elliptic curves $\mathbb{A}_g = \mathbb{B}_g$ *for all* g*.*

By coordinate change $x_k = \exp(i\pi z_k)$,

Computing Feynman integrals

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Proposition (BBBM '15)

$$P(x,q) = \frac{x^2}{(x^2-1)^2} + \sum_{a=1}^{\infty} \sum_{w|a} w(x^{2w} + x^{-2w})q^{2a}$$

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Sketch of Proof

$$\{\text{labeled tropical covers}\} \overset{1:1}{\leftrightarrows} \{\text{constant products of Laurent monomials}\}$$

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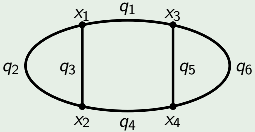
Example



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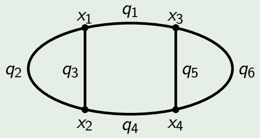

$x_1 < x_3 < x_4 < x_2 \qquad \underline{a} = (0, 2, 2, 0, 1, 0)$

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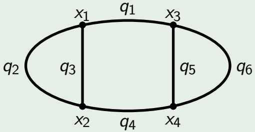

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$$\left(\frac{x_1}{x_3}\right)^2 \cdot 2 \cdot \left(\frac{x_2}{x_1}\right)^{2\cdot 2} \cdot \left(\frac{x_1}{x_2}\right)^2 \cdot \left(\frac{x_4}{x_2}\right)^2 \cdot \left(\frac{x_4}{x_3}\right)^2 \cdot 2 \cdot \left(\frac{x_3}{x_4}\right)^{2\cdot 2}$$

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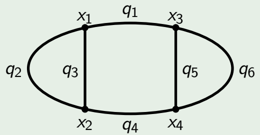

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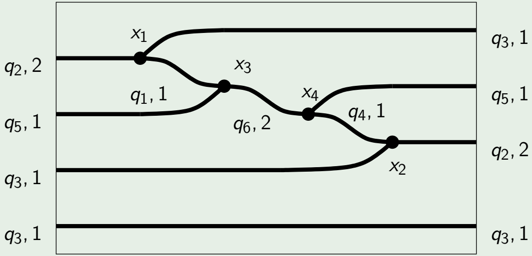

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