Towards a tropical interpretation of higher genus mirror symmetry

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joint work in progress with Arne Buchholz and Hannah Markwig

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19 September 2012

Calabi-Yau varieties and mirror symmetry

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Algebraic geometry of X^* $\leftrightarrow$ Symplectic geometry of X

Deformations of

complex structure symplectic structure

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Local isomorphism identifies tangent spaces

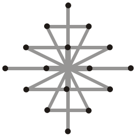
$$H^1(T_{X^*}) = H^{d-1,1}(X^*) \quad \cong \quad H^{1,1}(X)$$

by Bogomolov-Tian-Todorov if $\quad$ Moser

$0 = H^0(T_{X^*}) = H^{d-1,0}(X^*)$

Calabi-Yau varieties and mirror symmetry

$d = 3:$



$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
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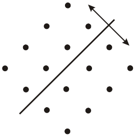

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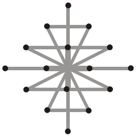
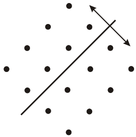


$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$



Local isomorphism $\to$ mirror map $\to$ equality of formal power series

$$\begin{aligned} \mathbb{B}_g^{X^*}(Q) &= \mathbb{A}_g^{X}(q) \\ Q \cdot e^{J(Q)} &= q \end{aligned}$$

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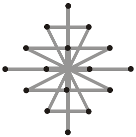

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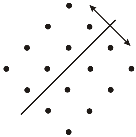

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Path integrals over trivalent Feynman graphs

Gromov-Witten theory of X

$\bar{M}_g(X, d) \ni C \rightarrow X$

(quantum cohomology if $g = 0$)

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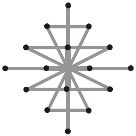

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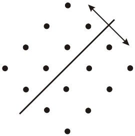

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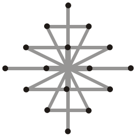

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0 $h^{1,1}$ 0
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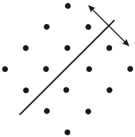

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On the level of tangent spaces: Interpret lattice points as
Deformations of X^* Divisor classes of X

Degenerations

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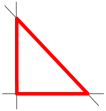

$$X_0 = \{x_0 x_1 x_2 = 0\} \subset \mathbb{P}^2$$
$$X_t \qquad + t \cdot [3]$$

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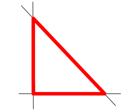

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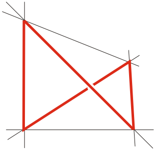

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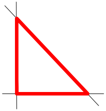

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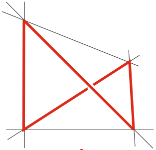

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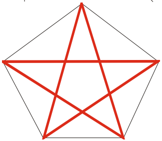

$$X_0 = \{x_0x_1 = x_1x_2 = x_2x_3 = x_3x_4 = x_4x_0 = 0\} \subset \mathbb{P}^4$$

X_t by structure theorem of Buchsbaum-Eisenbud

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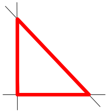

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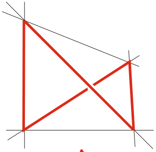

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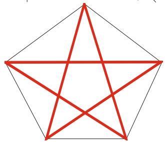

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X_t by structure theorem of Buchsbaum-Eisenbud

$\mathbb{Q}$-Gorenstein toric Fano Y	$\rightleftarrows$	$\Sigma =$ Fan over Fano polytope Δ^*
$\mathfrak{X} \to \operatorname{Spec}\mathbb{C}[t]$	$\rightleftarrows$	$I \subset \mathbb{C}[t] \otimes \operatorname{Cox}(\Sigma(1))$
Special fiber $X_0 \subset Y$		$I_0 \subset \operatorname{Cox}(\Sigma(1))$

Tropical mirror correspondence

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Associate to $\mathfrak{X}$ two metric complexes (abstract tropical varieties):

$\mathrm{Strata}(X_0) \subset \Delta = \mathrm{Strata}(Y)$ $\qquad$ $T_{I_0}(I) \subset \nabla_{I_0}(I) = \{w \mid L_w(I_0) = I\}$

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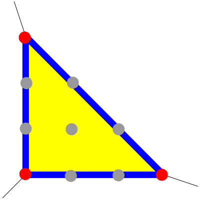

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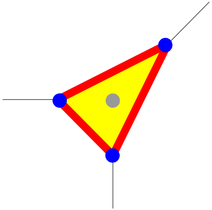

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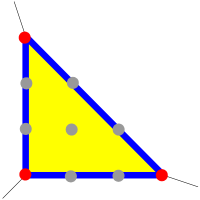

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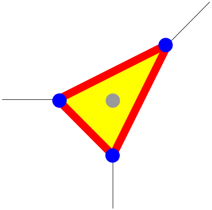

Hypersurfaces: $\nabla_{I_0}(I) = \Delta^*$ (dual of Newton-polytope) $\to$ Batyrev

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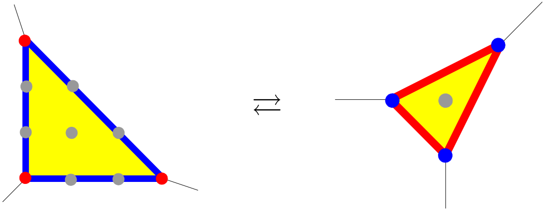

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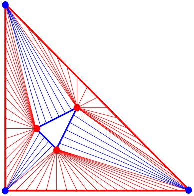
Limit of points in the tropical fiber: $\text{Strata}(X_0) \leftarrow T_{I_0}(I)$



Mirror symmetry for the elliptic curve

Complex structure modulus:

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$$\mathfrak{H} = \{s = s_1 + is_2 \in \mathbb{C} \mid s_2 > 0\} \qquad \Gamma = \mathrm{PSL}(2,\mathbb{Z}) \text{ modular group}$$

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$$E = \mathbb{C}/\Lambda_s \qquad \Lambda_s = 1\cdot\mathbb{Z} + s\cdot\mathbb{Z} \qquad s \in \mathfrak{H}$$

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Complexified Kähler class $[\omega] \in H^2(E, \mathbb{C})$ on E.

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Mirror symmetry:

$$\begin{array}{ccc} \text{Kähler modulus} & \longleftrightarrow & \text{complex modulus} \\ E_{s,t} & \longleftrightarrow & E_{t,s} \end{array}$$

Embedding to projective space

Weierstrass $\wp$-function with period lattice $\Lambda_\tau = 1 \cdot \mathbb{Z} + s \cdot \mathbb{Z}$

$$\wp(z,s) = \frac{1}{z^2} + \sum_{n^2+m^2 \neq 0} \left(\frac{1}{(z+m+ns)^2} - \frac{1}{(m+ns)^2} \right)$$

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with the Eisenstein series and divisor power sums

$$E_{2k}(Q) = 1 - \tfrac{2k}{B_k} \textstyle\sum_{n=1}^{\infty} \sigma_{k-1}(n) Q^n \qquad \sigma_{k-1}(n) = \textstyle\sum_{m|n} m^{k-1}$$

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Embedding to projective space (Weierstrass normal form)

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Modular forms $= \mathbb{C}[E_4, E_6] \subset \mathbb{C}[E_2, E_4, E_6] =$ quasimodular forms.

A-model: Hurwitz numbers

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Definition

Fix $2g-2$ distinct points $B = \{p_1, \ldots, p_{2g-2}\} \subset E$.

$$\mathrm{Cov}_{d,g}(E,B) = \left\{ \begin{array}{l} C \to E \text{ degree } d \text{ simple branched at } B \\ C \text{ irreducible genus } g \text{ curve} \end{array} \right\} / \sim$$

$$(\pi : C \to E) \sim (\pi' : C' \to E) \Leftrightarrow \exists \phi : C \tilde{\to} C' \text{ with } \pi' \circ \phi = \pi.$$

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With $\mathrm{Aut}(\pi)=\{\phi:C\xrightarrow{\sim}C\mid \pi\circ\phi=\pi\}$ define

$$N_{g,d}=\sum_{\pi\in\mathrm{Cov}_{d,g}(E,B)}\frac{1}{\#\,\mathrm{Aut}(\pi)}$$

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With $\mathrm{Aut}(\pi)=\{\phi:C\xrightarrow{\sim}C\mid\pi\circ\phi=\pi\}$ define

$$N_{g,d}=\sum\nolimits_{\pi\in\mathrm{Cov}_{d,g}(E,B)}\frac{1}{\#\,\mathrm{Aut}(\pi)}$$

In disconnected case $C=\bigcup_i C_i$ analogously $\widehat{\mathrm{Cov}}_{g,d}(E,B)$ and $\hat{N}_{g,d}$.

Computing Hurwitz numbers for any genus and degree

Fix $p_0 \notin B$ as basepoint. $\widehat{\mathrm{Cov}}_{g,d}(E,B)' = \left\{\pi \text{ with marking of } \pi^{-1}(p_0)\right\}$

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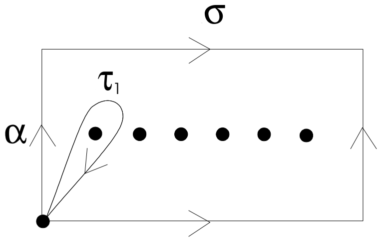
Monodromy map

$$\mathrm{mon} : \widehat{\mathrm{Cov}}_{g,d}(E,B)' \to \hat{T}_{g,d}$$



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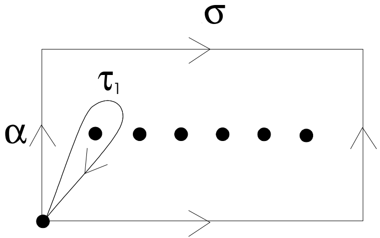

Lemma

$$\widehat{\mathrm{Cov}}_{g,d}(E,B) = \widehat{\mathrm{Cov}}_{g,d}(E,B)'/S_d \xrightarrow{1:1} \hat{T}_{g,d}/S_d$$

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Theorem (Roth, Yui)

$$\hat{N}_{g,d} = \mathrm{Tr}\left(M(d)^{2g-2}\right)$$

Generating functions

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Conjugacy class basis vs representation basis

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Center $\mathcal{H}_d \subset \mathbb{C}[S_d]$ has dimension $\mathrm{part}(d)$ and bases

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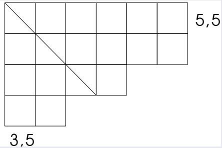

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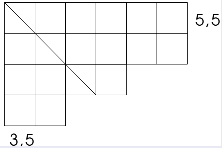

Theorem (Douglas)

$$\hat{Z}(q,\lambda) = -1 + \mathrm{coeff}_{w^0} \prod_{u \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left(1 + w q^u e^{\frac{u^2}{2}\lambda}\right) \prod_{v \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left(1 + w^{-1} q^v e^{\frac{-v^2}{2}\lambda}\right)$$

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$$\mathbb{C}[E_2, E_4, E_6] = \mathbb{C}[E_2, E_2', E_2'']$$

Example

$$F_2 = -\frac{1}{720}\cdot(E_2E_2' + E_2'') \qquad F_3 = \frac{1}{20736}\cdot(7(E_2')^3 + 3(E_2'')^2)$$

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$F_g(e^{\pi i t})$ are (meromorphic limit of) a (non-homomorphic) section of $L^{\otimes(2g-2)}$ where L is the line bundle

$$\mathcal{M}_{E^*} \ni t \leftarrow H^0(E_t^*, K_{E_t^*})$$

and hence should be quasimodular.

Tropical Hurwitz numbers

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Definition

Branch map $\mathrm{br}^{trop}: \begin{array}{ccc} M_g^{trop}(S^1, d) & \longrightarrow & (S^1)^{2g-2} \\ (\pi : \Gamma \to S^1) & \mapsto & (\pi(p_1), \ldots, \pi(p_{2g-2})) \end{array}$

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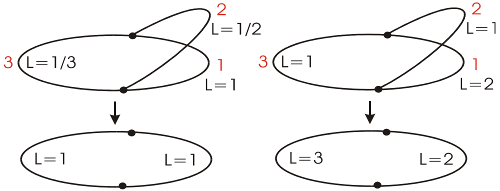

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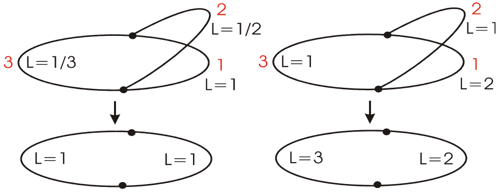

Leads to combinatorial rule (or definition):

Theorem

$H_{g,d}^{trop} = \sum_\Gamma \mathrm{mult}(\Gamma)$ *where* Γ *is genus* g*, weighted, balanced, trivalent graph with* $d : 1$ *map to* S^1 *and* $2g-2$ *branch points with cyclic labelling.*

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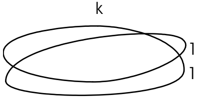
$\mathrm{mult}(\Gamma) = \frac{1}{\#\,\mathrm{Aut}(\Gamma)} \prod_{e \in \mathrm{e}(\Gamma)} w(e)$

mult 

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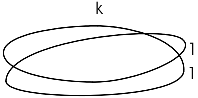
mult 

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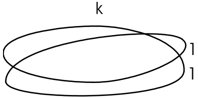
mult  $= \frac{1}{k}$

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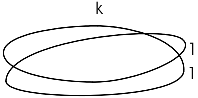
mult  $= \frac{1}{k}$

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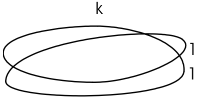
mult 

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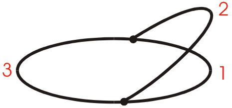

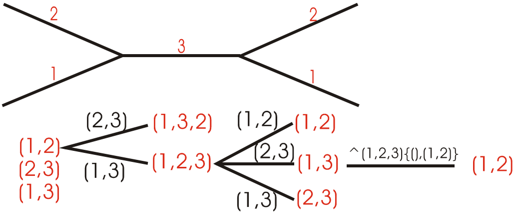

$\frac{1}{3!} \cdot 3 \quad \cdot \quad 2 \quad \cdot \quad 3 \quad \cdot \quad 2 = 6$

Tropical Hurwitz numbers – Example

$$\hat{N}_{3,3} = ?$$

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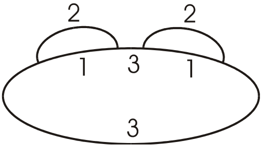
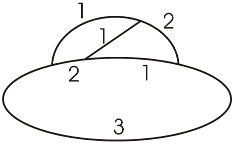
$2\cdot$



$4\cdot$



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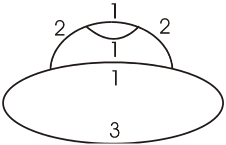

$\text{mult}(\Gamma) = 2^2 \cdot 3^2 = 36$

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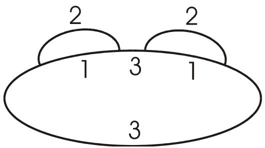
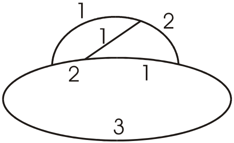
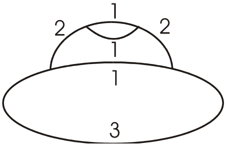
$2\cdot$ 

$4\cdot$ 

$4\cdot$ 

$\text{mult}(\Gamma) = 2^2 \cdot 3^2 = 36$ $\quad$ $\text{mult}(\Gamma) = 2^2 \cdot 3 = 12$

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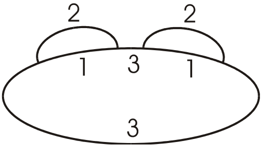

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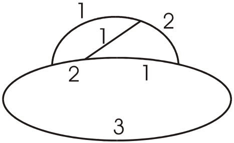

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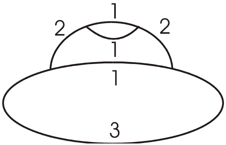

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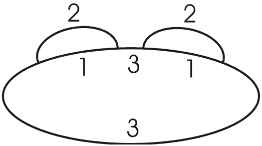

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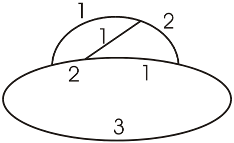

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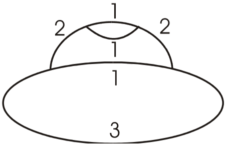

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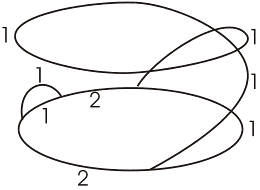

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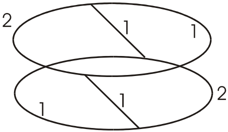

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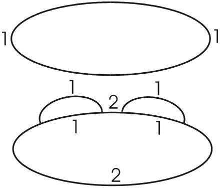

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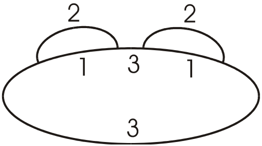

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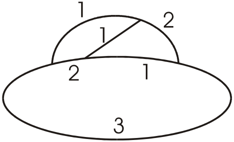

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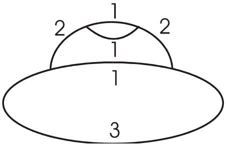

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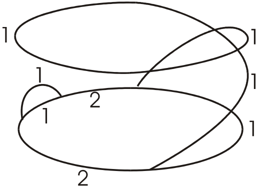

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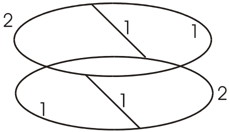

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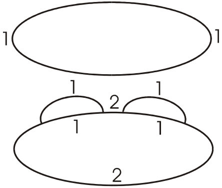

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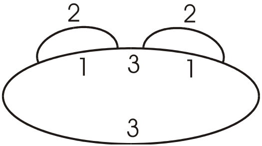

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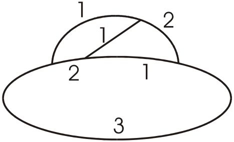

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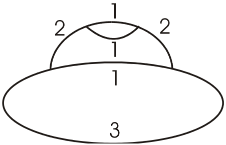

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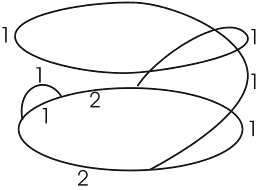

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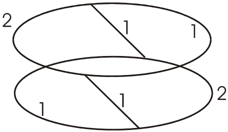

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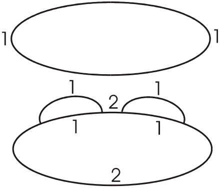

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Tropical Hurwitz numbers – Example

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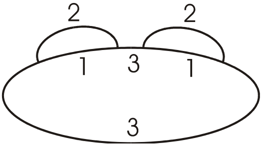

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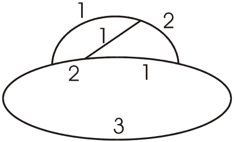

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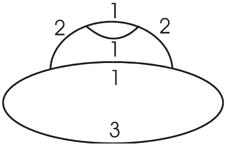

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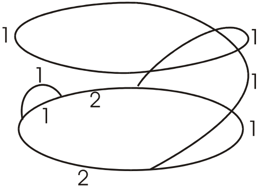

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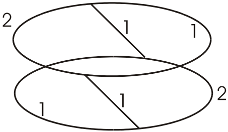

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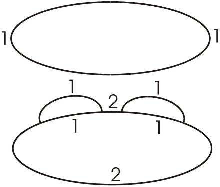

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Tropical computation of finer invariants

Theorem

$$\left(M(d)^{2g-2}\right)_{[\sigma],[\sigma]} = \sum_{\Gamma \text{ with partition } [\sigma] \text{ over } \overline{p_1 p_2}} \operatorname{mult}(\Gamma)$$

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Example

$$M(3)^{2\cdot 3-2} = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}^4 = \begin{pmatrix} 27 & 0 & 27 \\ 0 & 81 & 0 \\ 54 & 0 & 54 \end{pmatrix} \quad \begin{matrix} 1+1+1 \\ 1+2 \\ 3 \end{matrix} \qquad \mathrm{Tr} = 162$$

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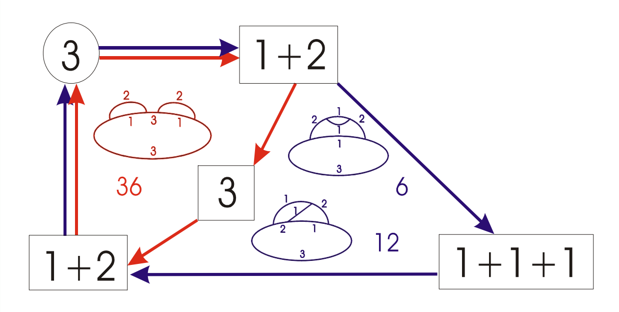

B-model: Feynman integrals

Basic idea: Count constant maps $f : C \longrightarrow E^*$.

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$$P(z) = \begin{cases} \frac{1}{4\pi^2}\wp(q,z) + \frac{1}{12}E_2(q) & \text{if } z \neq 0 \\ \frac{1}{12}E_2(q) & \text{if } z = 0 \end{cases}$$

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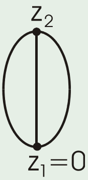
$$F_g = \sum_{\Gamma} \frac{1}{\#\operatorname{Aut}(\Gamma)} \int_{z_1} dz_1 \ldots \int_{z_{2g-2}} dz_{2g-2} \prod_{\{v,w\} \in \mathrm{e}(\Gamma)} P(\pi_z(w) - \pi_z(v))$$

where π_z varies over all positions z_i of branch points.

Feynman integrals

Example (Genus 2)



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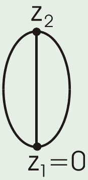

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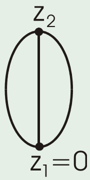

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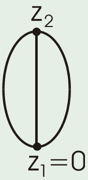

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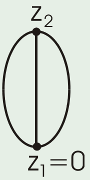

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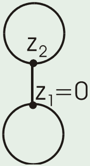

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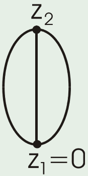

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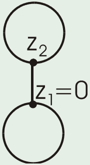

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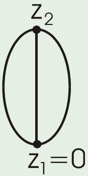

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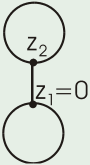

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This matches nicely tropical geometry: The second graph does not lead to tropical covers, since the weight 0 edge would be contracted.