Tropical Mirror Symmetry for Elliptic Curves

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Outline

- Mirror theorems
- Hurwitz numbers
- Feynman integrals
- Mirror symmetry for elliptic curves

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- Tropical Hurwitz numbers
- Correspondence theorem
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- Quasimodularity
- Computational point of view

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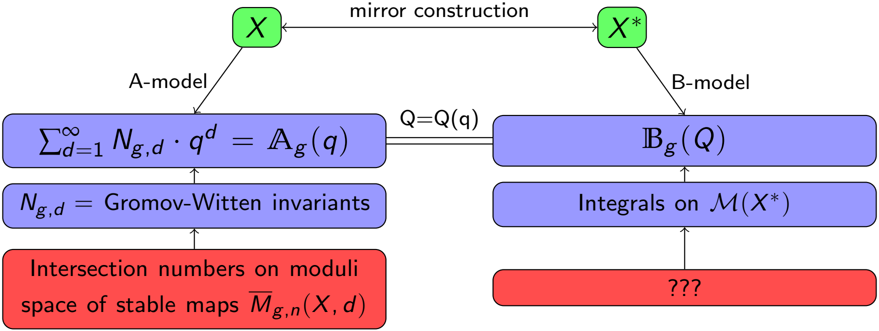

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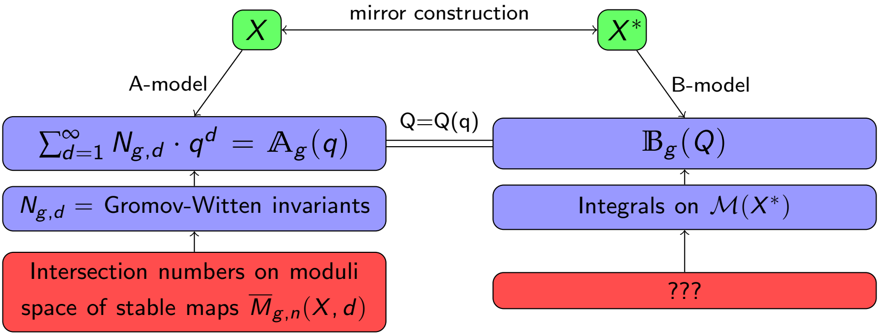

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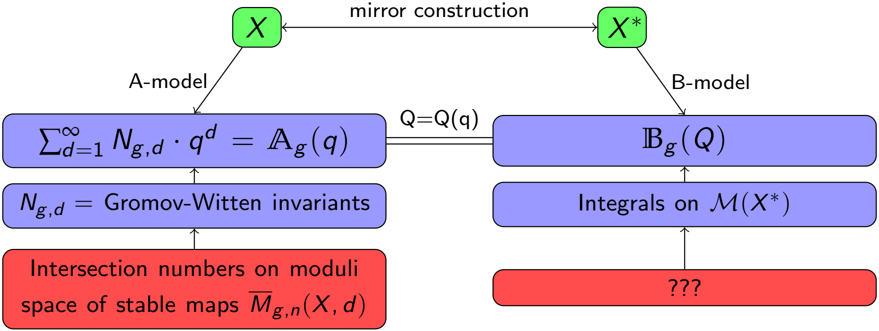

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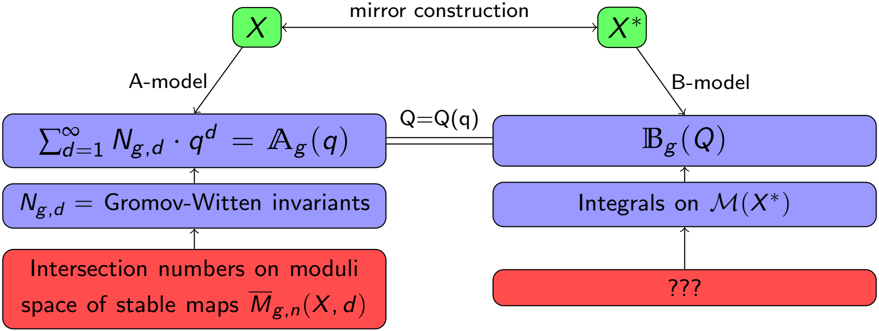

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- Algebraic/symplectic geometry: Fulton-Pandharipande '95, Kontsevich '95, Behrend-Fantechi '97,...

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- What are the B-model integrals?

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Hurwitz numbers are the Gromov-Witten invariants in A-model:

Theorem (special case of Okounkov-Pandharipande '06)

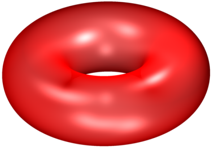
$$N_{g,d} = \int_{[\overline{M}_{g,2g-2}(E,d)]} \psi_1 \, \mathrm{ev}_1^*(x_1) \cdot \ldots \cdot \psi_{2g-2} \, \mathrm{ev}_{2g-2}^*(p_{2g-2})$$

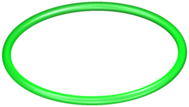
with Psi-classes $\psi_i = \mathrm{ch}_{top}\left(\Omega^1_{C,x_i} \mapsto (C, x_1, \ldots, x_{2g-2}, f)\right)$.

Tropical point of view

How to understand *all* $N_{g,d}$? Pass to **tropical geometry**:

$$E \qquad \mapsto \qquad \mathrm{trop}(E)$$





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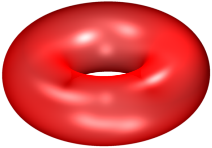

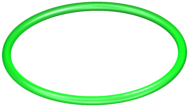

$$\begin{tikzcd}
\text{Gromov-Witten invariants} \arrow[rr, leftrightarrow, "\text{Mirror Theorem}"] & & B\text{-model} \\
& \text{tropical Gromov-Witten invariants} \arrow[ul, leftrightarrow, "\text{Correspondence Theorem}"] \arrow[ur, leftrightarrow, "\text{Tropical Mirror Theorem}"'] &
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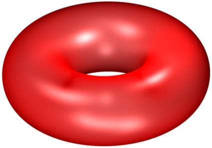

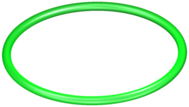

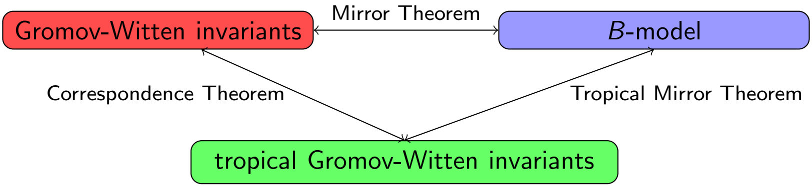

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- tropical mirror theorem (Gross '10)

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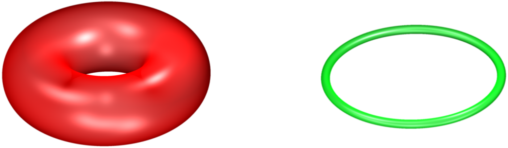

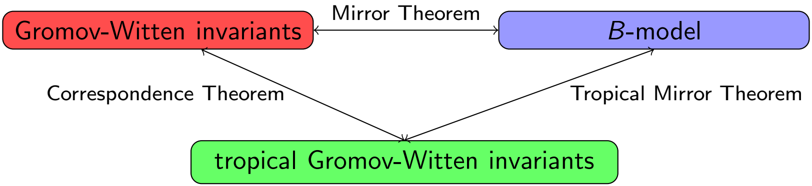

For $X = \mathbb{P}^2$ (building block of C-Y) and $g = 0$:

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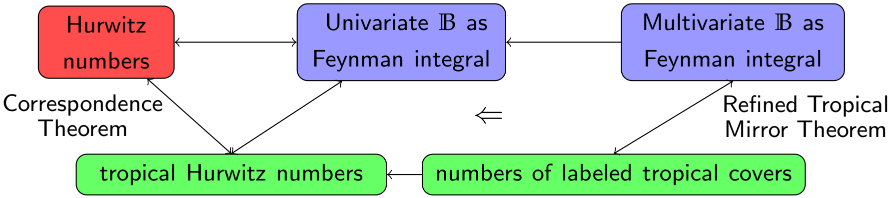
- partial correspondence theorem (Markwig-Rau '09)

Our results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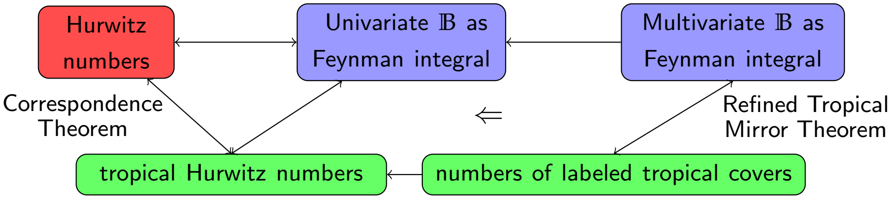

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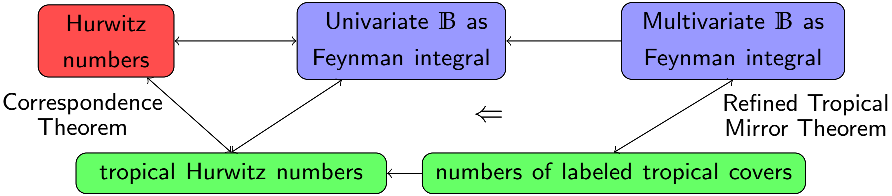

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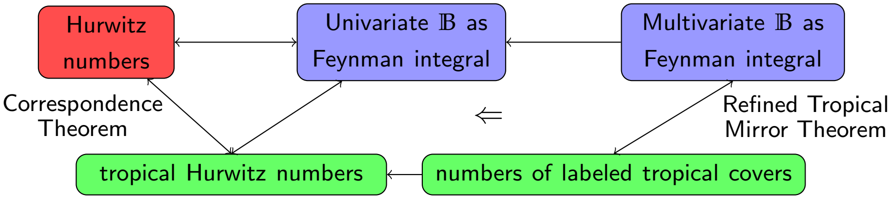

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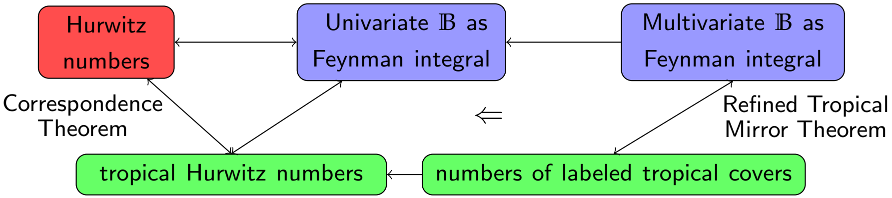

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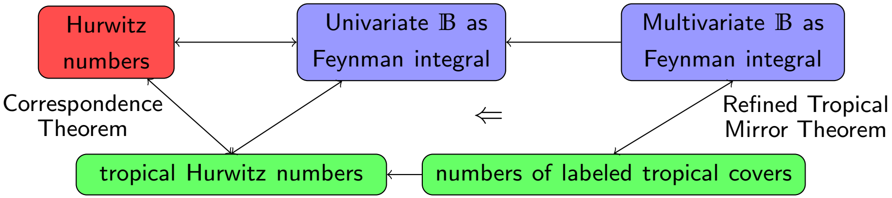

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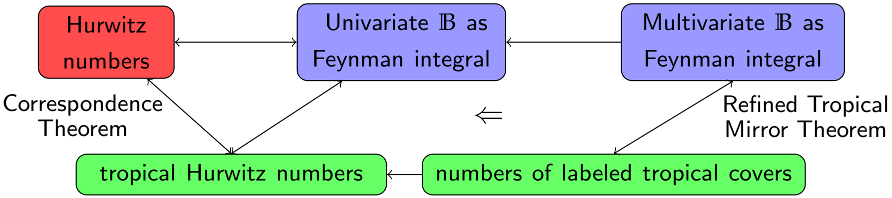

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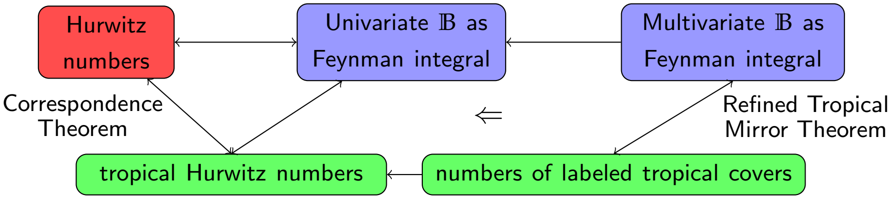

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- Implications in number theory: refined generating functions are quasi-modular.

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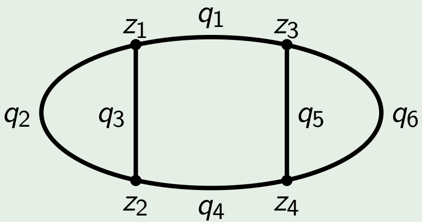

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$E_2 = 1 - 24\sum_{d=1}^{\infty}\sigma_1(d)q^{2d} = 1 - 24q^2 - 72q^4 - \ldots \qquad \sigma_1(d) = \sum_{m|d} m$

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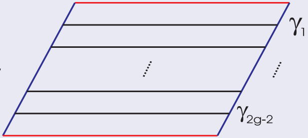
Definition (Feynman integral)

For ordering $\Omega \in S_{2g-2}$ of integration paths on E



$$I_{\Gamma,\Omega} = \int_{\gamma_{2g-2}} \ldots \int_{\gamma_1} \left(\prod_{e \in \mathrm{edges}(\Gamma)} P_k(z_e^+ - z_e^-, q) \right) dz_{\Omega(1)} \ldots dz_{\Omega(2g-2)}$$

Mirror symmetry for elliptic curves

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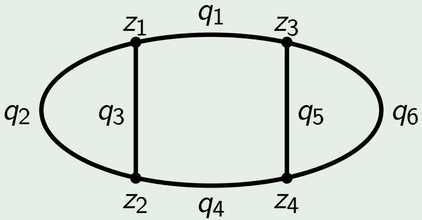

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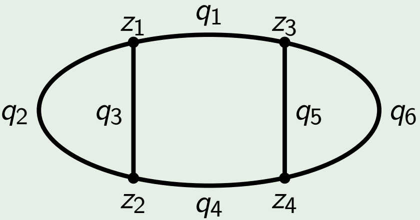

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Theorem (Dijkgraaf '96)

For $g > 1$

$$\sum_d N_{g,d}\ q^{2d} = \sum_{g(\Gamma)=g} \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\Omega} I_{\Gamma,\Omega}(q)$$

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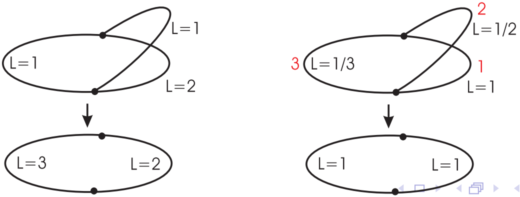
Tropical covers are balanced w.r.t. weights $w(e)$:



Correspondence Theorem

As a direct generalization of (Cavalieri-Johnson-Markwig '10) and (Bertrand-Brugallé-Mikhalkin '11) obtain correspondence theorem:

Correspondence Theorem

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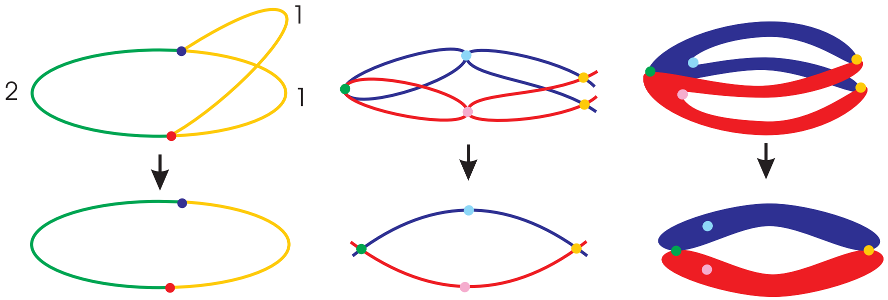

Tropical Hurwitz numbers – Example

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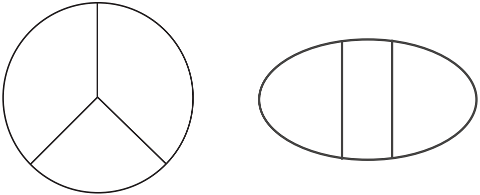
Two trivalent, connected combinatorial types (non-metric graphs)



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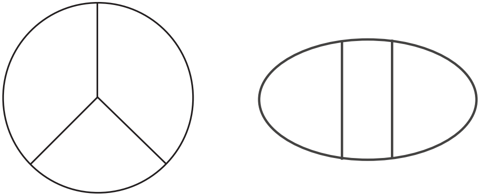

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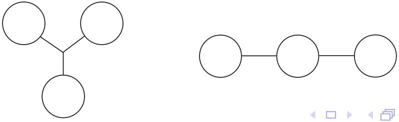

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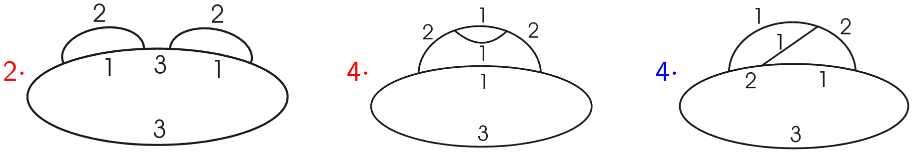

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Tropical Hurwitz numbers – Example

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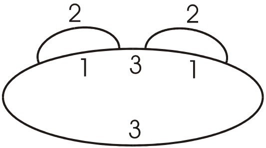
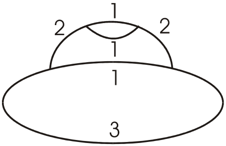
$2\cdot$



$4\cdot$



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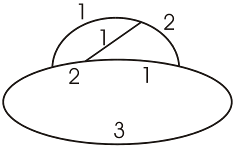

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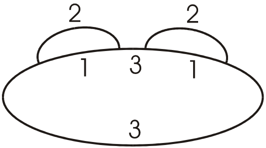

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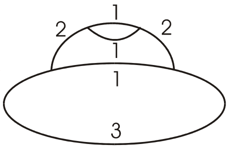

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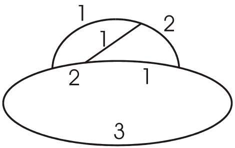

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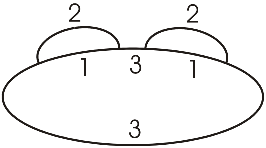

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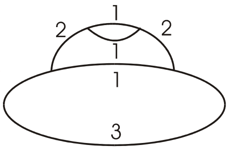

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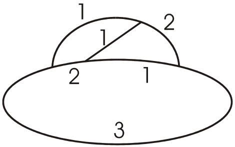

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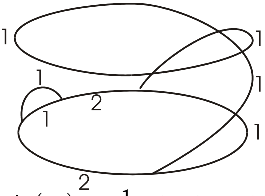

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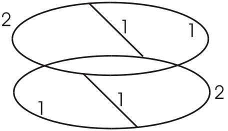

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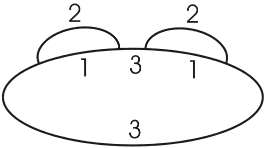

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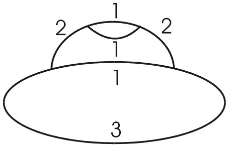

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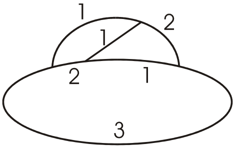

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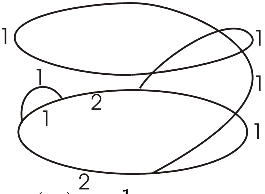

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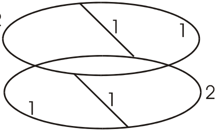

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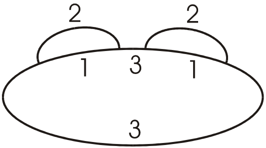
Tropical Hurwitz numbers – Example

$$N_{3,3}^{trop} = 112 + 48 = 160$$

2·



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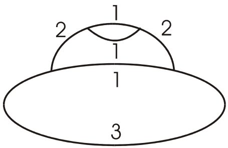

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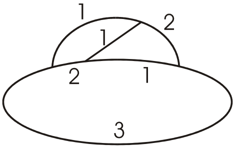

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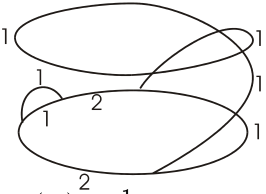

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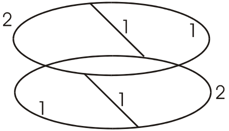

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Labeled tropical covers (refined A-side)

Fix a base point $p_0 \in E$.

Let Γ be a Feynman graph, $\underline{a} = (a_1, ..., a_{3g-3}) \in \mathbb{N}^{3g-3}$, and $\Omega \in S_{2g-2}$.

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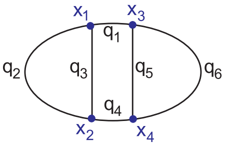
counted with multiplicity

$$\mathrm{mult}(\pi) = \prod_{e \in \mathrm{edges}(C)} w(e)$$

Example

$\underline{a} = (0, 1, 1, 0, 1, 0)$ $\Gamma =$ 

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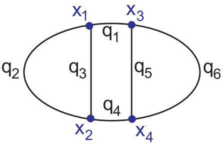
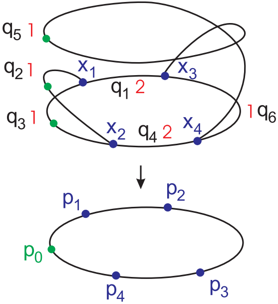


$$\Omega = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$



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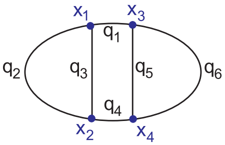

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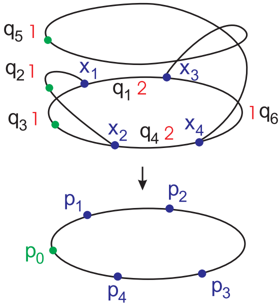

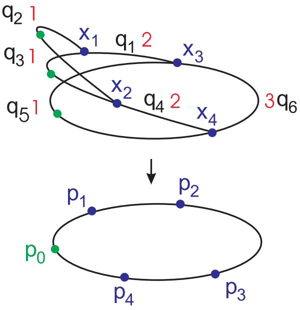

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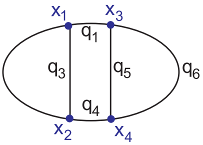
$\underline{a} = (0,1,1,0,1,0)$ $\quad \Gamma =$ 

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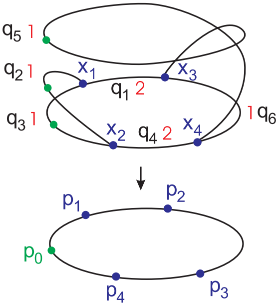

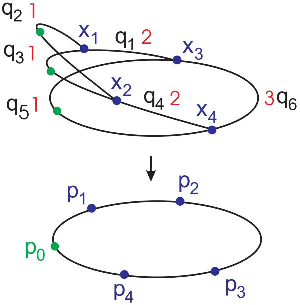

$$N^{trop}_{a,\Gamma,\Omega} = \quad 4 \quad + \quad 12 \quad = 16$$

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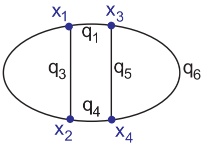
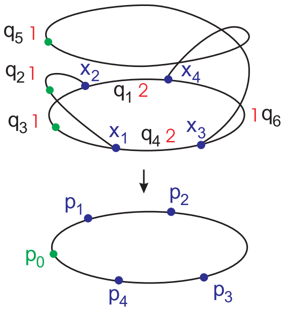
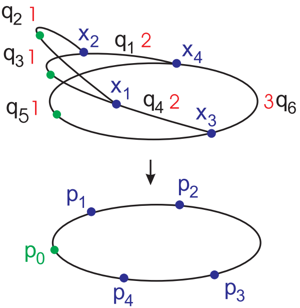


$\Omega = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$





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Refined Feynman integrals

Definition (Refined Feynman integrals)

$$I_{\Gamma,\Omega}(q_1, ..., q_{3g-3}) = \int_{\gamma_{2g-2}} ... \int_{\gamma_1} \left(\prod_{k=1}^{3g-3} P_k(z_k^+ - z_k^-, q_k) \right) dz_{\Omega(1)} ... dz_{\Omega(2g-2)}$$

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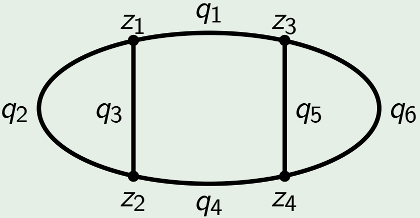

we have to integrate

$$P(z_1 - z_2, q_1) \cdot P(z_1 - z_2, q_2) \cdot P(z_1 - z_3, q_3) \cdot P(z_2 - z_4, q_4) \cdot P(z_3 - z_4, q_5) \cdot P(z_3 - z_4, q_6)$$

Tropical mirror theorem

Theorem (Multivariate tropical mirror theorem, BBBM '13)

$$\sum_{\underline{a}} N^{trop}_{\underline{a},\Gamma,\Omega}\ q^{2\underline{a}} = I_{\Gamma,\Omega}(q_1, \ldots, q_{3g-3})$$

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Setting $q_i = q$ we get:

Corollary (Tropical mirror theorem)

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Together with the correspondence theorem this proves:

Corollary (Mirror symmetry for elliptic curves)

For elliptic curves $\mathbb{A}_g = \mathbb{B}_g$ for all g.

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By coordinate change $x_k = \exp(i\pi z_k)$,

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Corollary

$$N^{trop}_{\underline{a},\Gamma,\Omega} = \text{const}_{x_{\Omega(2g-2)}} \ldots \text{const}_{x_{\Omega(1)}} \prod_{k=1}^{3g-3} P_{a_k}(x_k^+, x_k^-, q_k)$$

Implementation of Feynman integrals in Singular

Example

```
                     SINGULAR                                 /   Development
 A Computer Algebra System for Polynomial Computations       /    version 4
                                                           0<
 by: W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann     \    Dec 2013
 FB Mathematik der Universitaet, D-67653 Kaiserslautern       \
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Implementation of Feynman integrals in Singular

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                     SINGULAR                                 /   Development
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 by:  W. Decker, G.-M. Greuel, G. Pfister, H. Schoenemann    \    Dec 2013
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Implementation of Feynman integrals in Singular

Example

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  Graph with 4 vertices and 6 edges
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> generatingFunction (Gamma,2);
  8*q(1)^2+8*q(2)*q(3)+8*q(4)^2+8*q(5)*q(6)
```

Quasi-modularity

Corollary (BBBM '13, generalization of Kaneko-Zagier '95)

For all Feynman graphs Γ of genus g and all orders Ω the function $I_{\Gamma,\Omega}$ is a quasi-modular form ($I_{\Gamma,\Omega} \in \mathbb{Q}[E_2, E_4, E_6]$) of weight $6g-6$.

Eisenstein series $E_{2k} = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^{2n}$ $\qquad \sigma_{k-1}(n) = \sum_{m|n} m^{k-1}$

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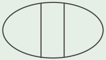
$E_4 = 1 + 240q^2 + 2160q^4 + \ldots \qquad E_6 = 1 - 504q^2 - 16632q^4 - \ldots$

Example

For $\Gamma =$  Singular gives

$$I_\Gamma = 32q^4 + 1792q^6 + 25344q^8 + 182272q^{10} + 886656q^{12} + O(q^{14})$$

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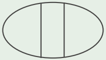
For $\Gamma =$  Singular gives

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$$I_\Gamma = \frac{16}{1492992}\left(4E_6^2 + 4E_4^3 - 12E_2E_4E_6 - 3E_2^2E_4^2 + 4E_2^3E_6 + 6E_2^4E_4 - 3E_2^6\right).$$

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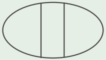
For $\Gamma =$  Singular gives

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$\Rightarrow$ Can compute $I_\Gamma(q)$ fast up to arbitrary high order.

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