Algebraic geometry, Gröbner bases, and Normalization

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14.10.2014

Algebraic Varieties

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Let K be a field. An **affine algebraic variety** is the common zero-set

$$V(f_1, ..., f_r) = \{p \in K^n \mid f_1(p) = 0, ..., f_r(p) = 0\}$$

of polynomials $f_1, ..., f_r \in K[x_1, ..., x_n]$.

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- the graph

$$\Gamma(g) = V\left(x_2 \cdot b(x_1) - a(x_1)\right) \subset K^2$$

  of a rational function

$$g = \frac{a}{b} \in K(x_1)$$

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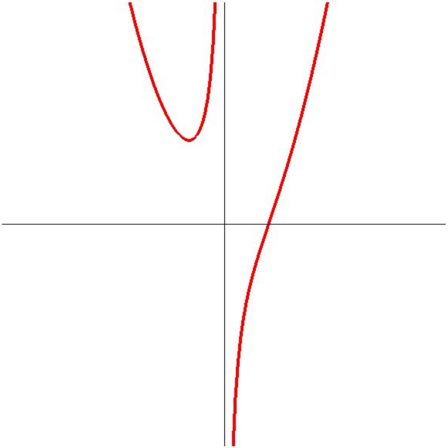

Examples of Varieties: Curves

Not every curve in K^2 is a graph, e.g., the **elliptic curve**

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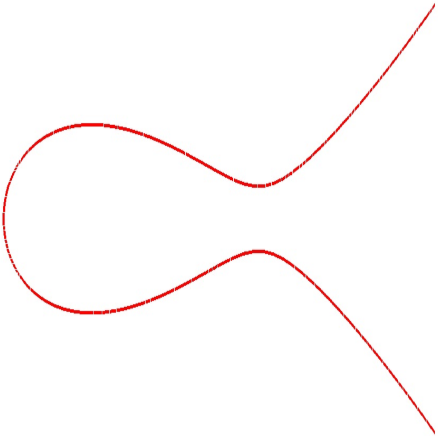

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Elliptic curves come with a group structure. They play an important role in Number Theory and Cryptography (e.g. Diffie-Hellman key exchange).

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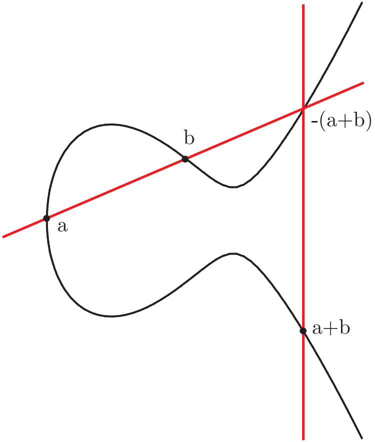

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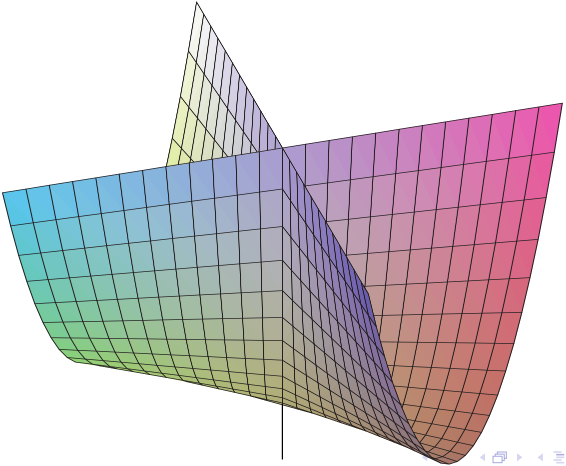

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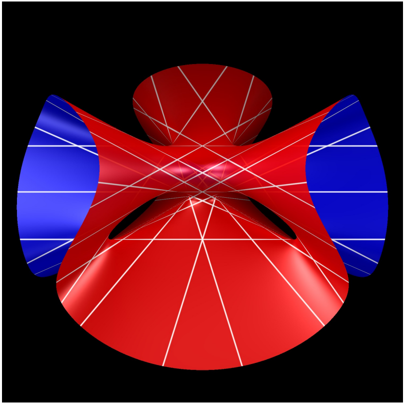

Examples of Varieties: Splines

The cubic spline $C \subset K^2$ parametrized by

$$x_1(t) = p_0(1-t)^3 + 3p_1 t(1-t)^2 + 3p_2 t^2(1-t) + p_3 t^3$$
$$x_2(t) = q_0(1-t)^3 + 3q_1 t(1-t)^2 + 3q_2 t^2(1-t) + q_3 t^3$$

with $t \in [0,1]$ passes through the points $(p_0, q_0), (p_3, q_3) \in K^2$ and the tangents at these points through (p_1, q_1) and (p_2, q_2).

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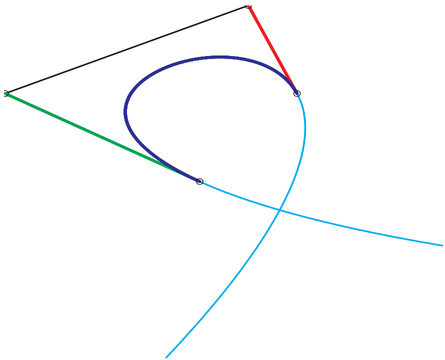

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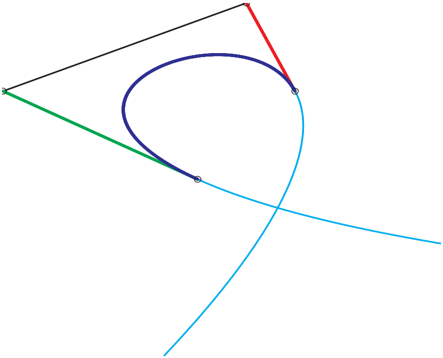

The curve sector C is a subset of an algebraic curve $\overline{C}$.

$\overline{C}$ is the closure of C in the **Zariski Topology**, which has as closed sets the algebraic varieties.

Projections

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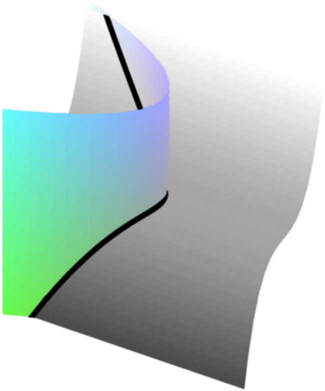

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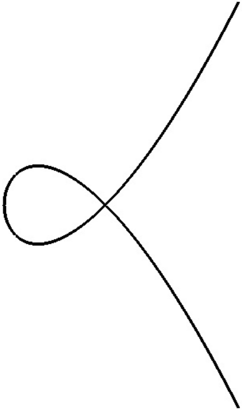

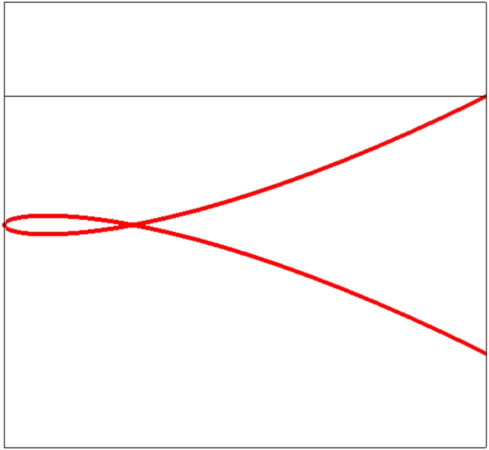

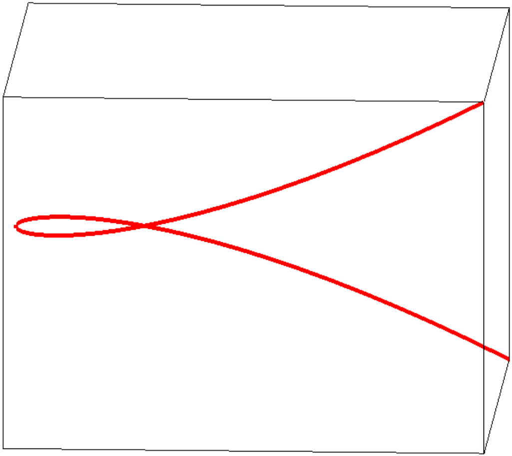

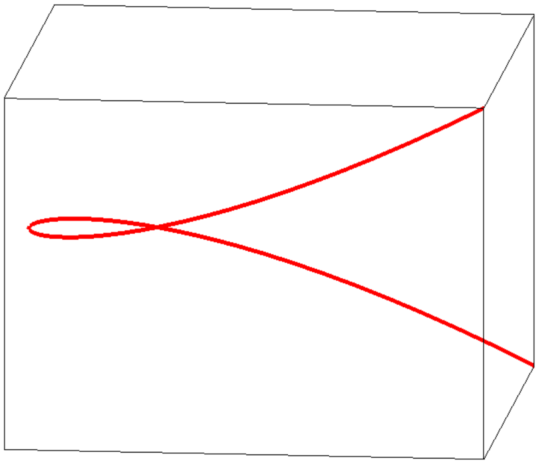

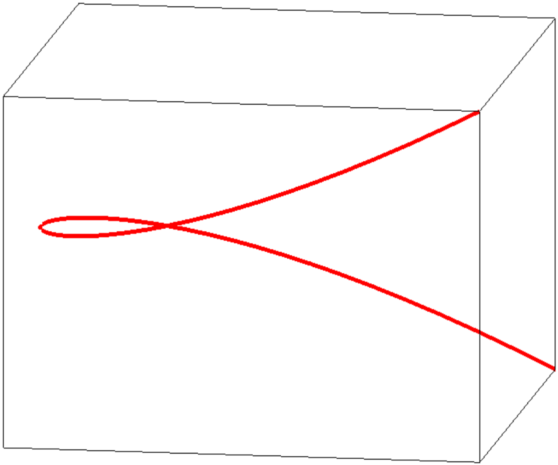

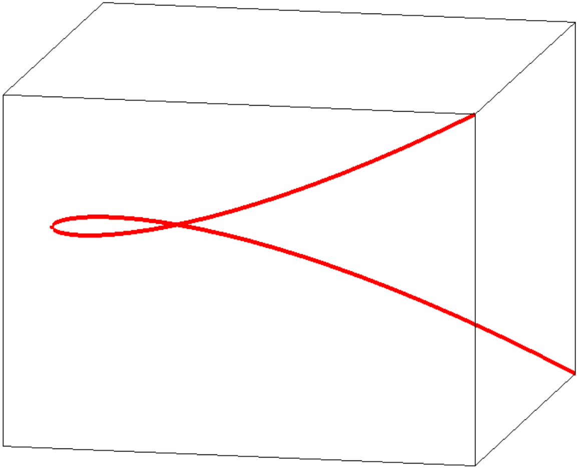

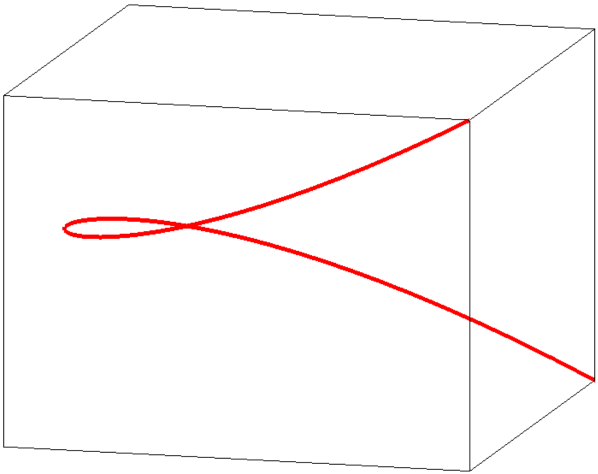

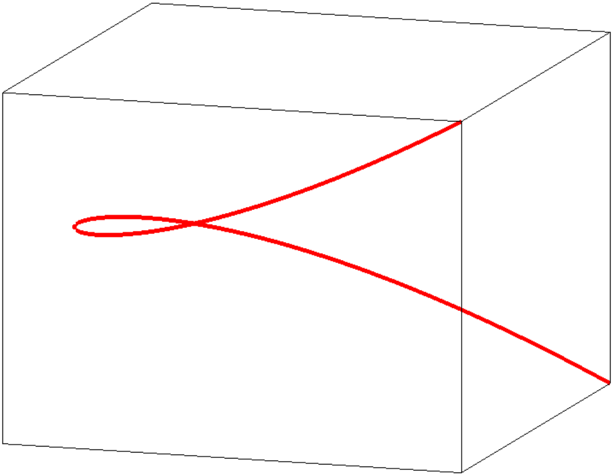

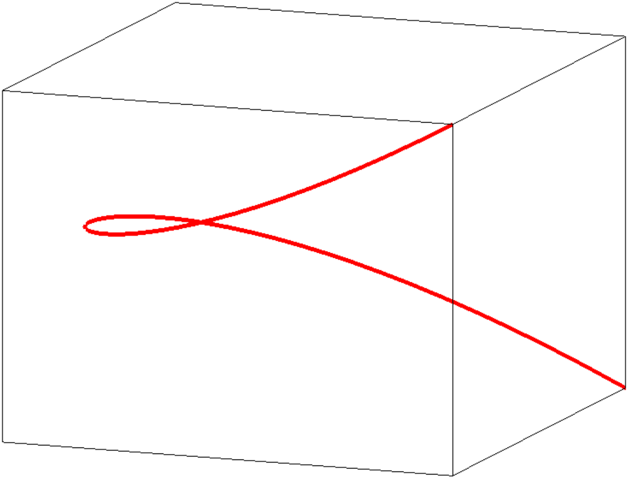

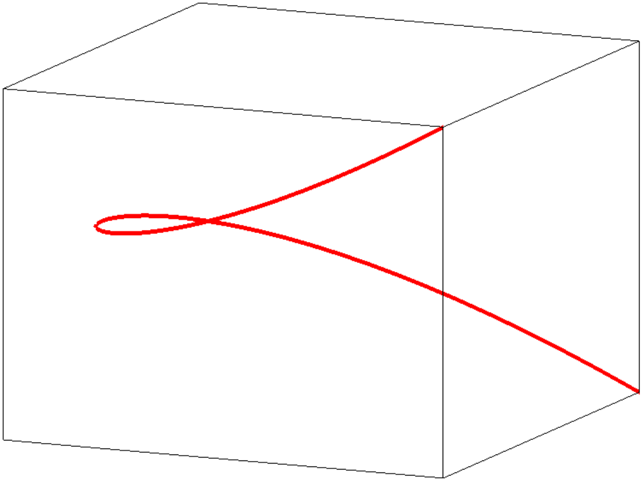

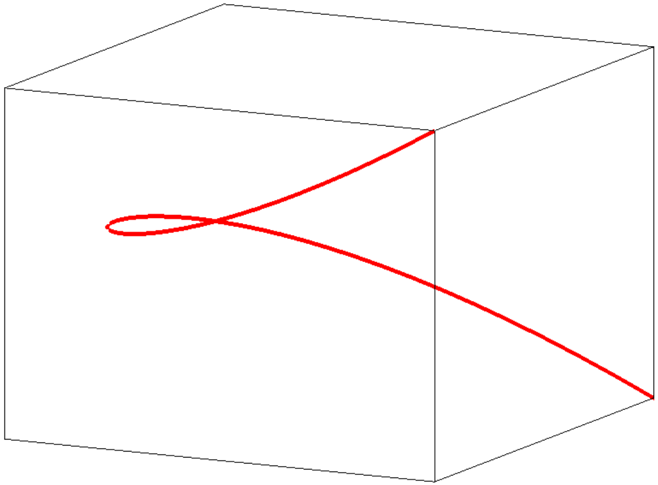

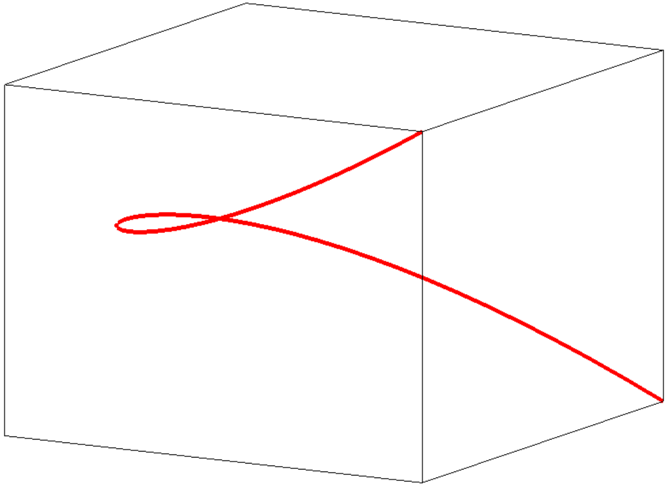

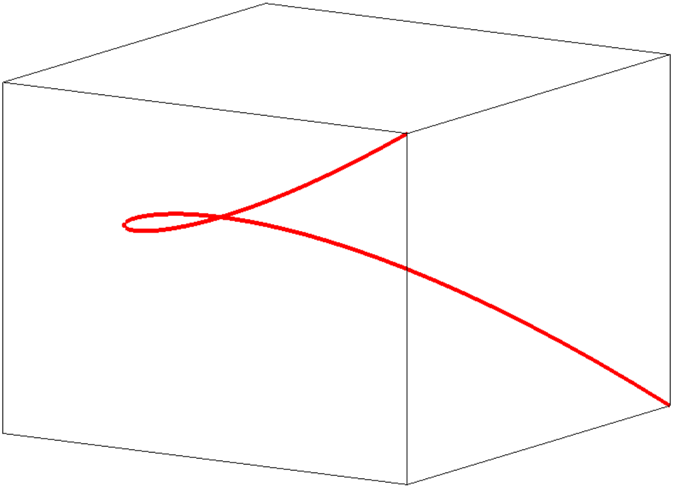

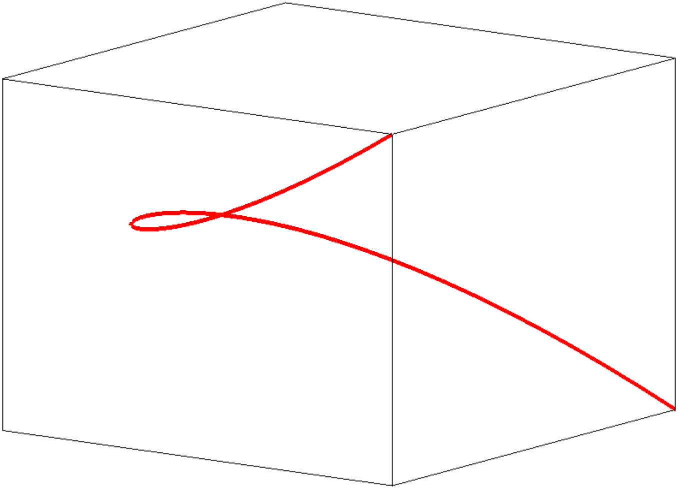

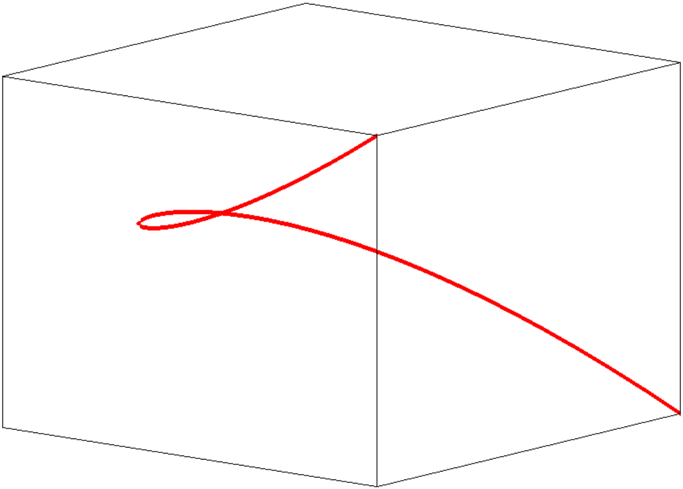

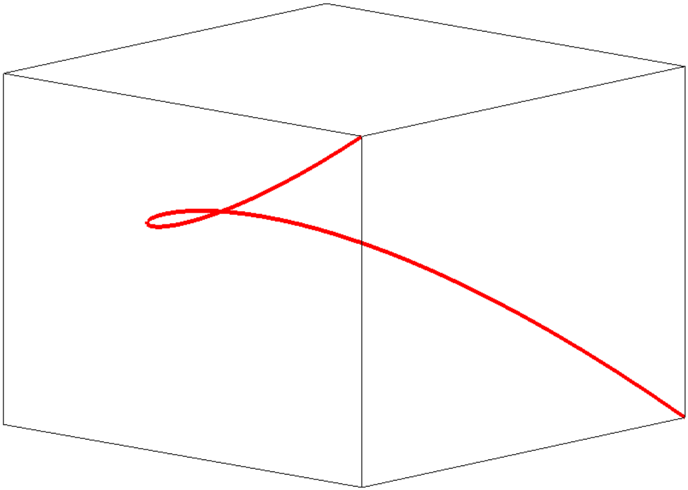

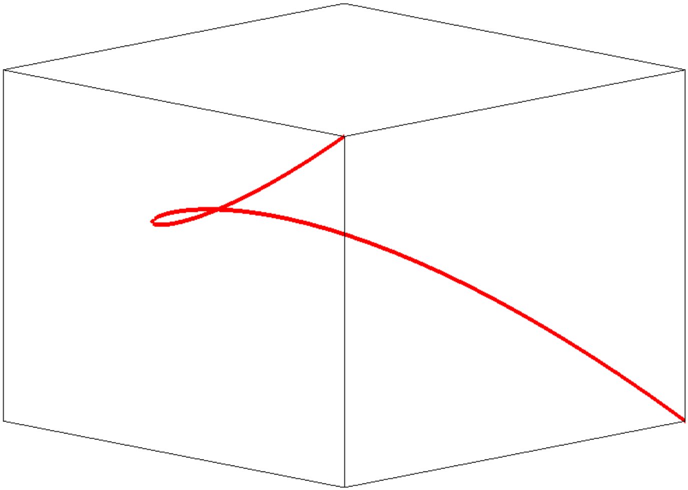

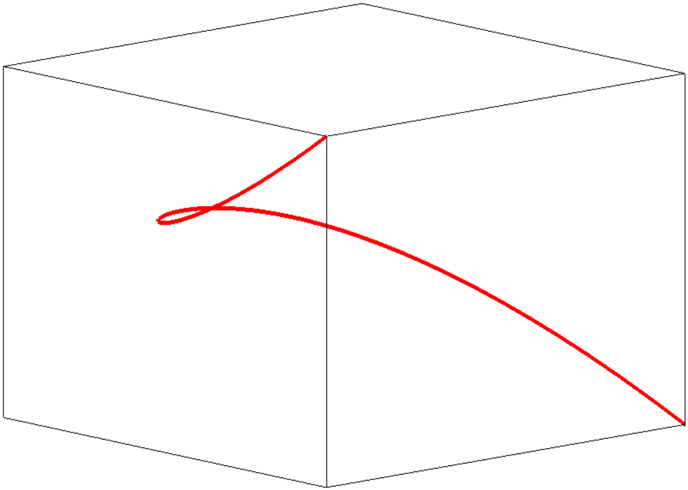

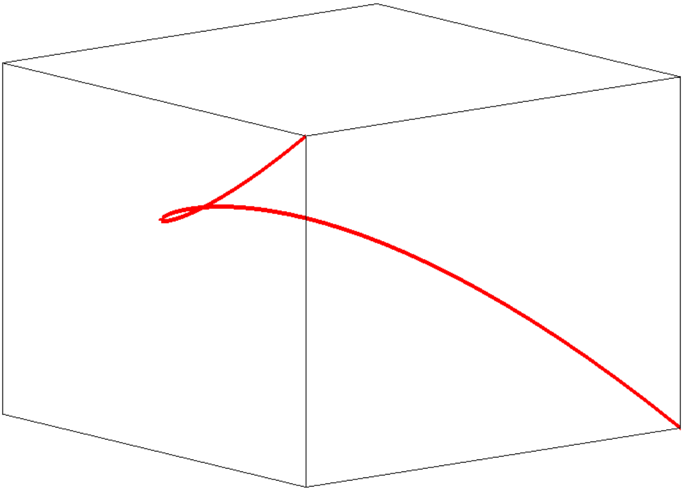

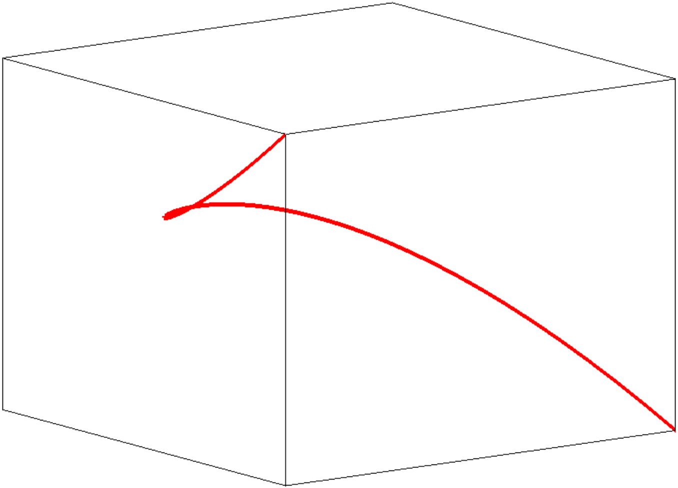

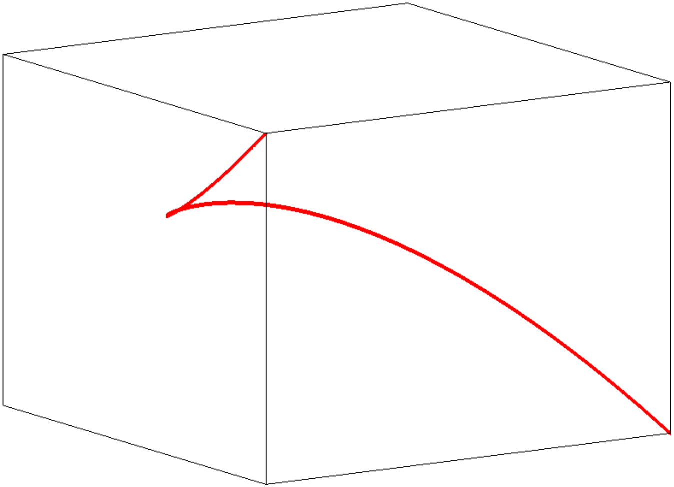

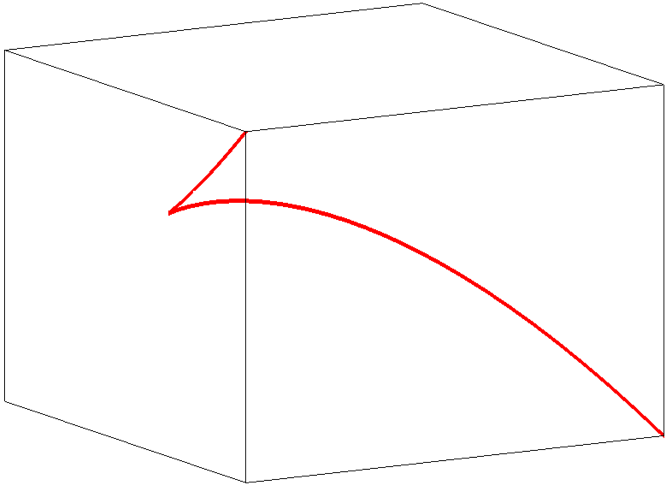

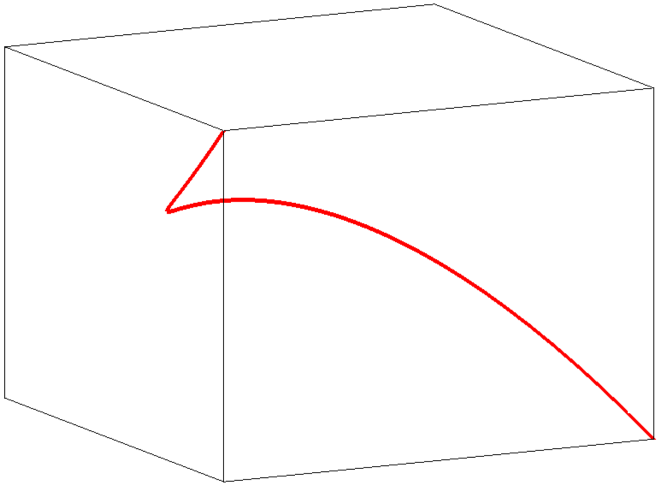

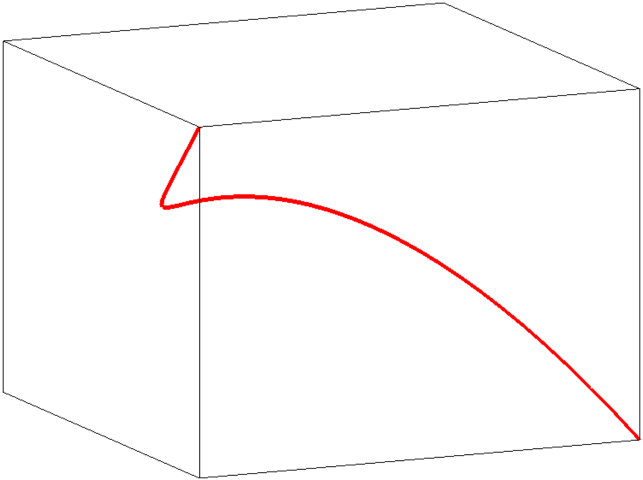

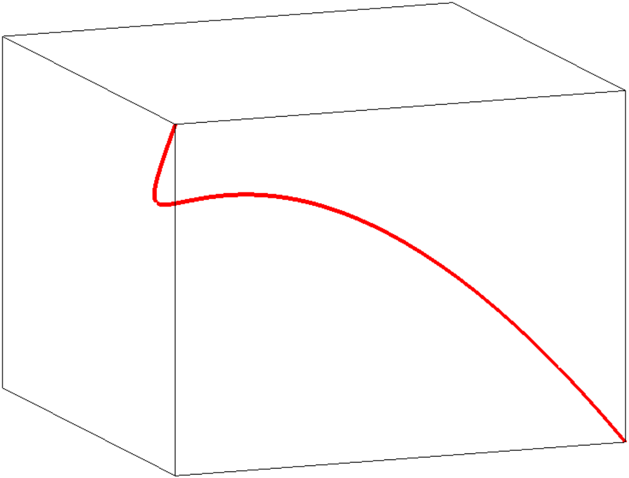

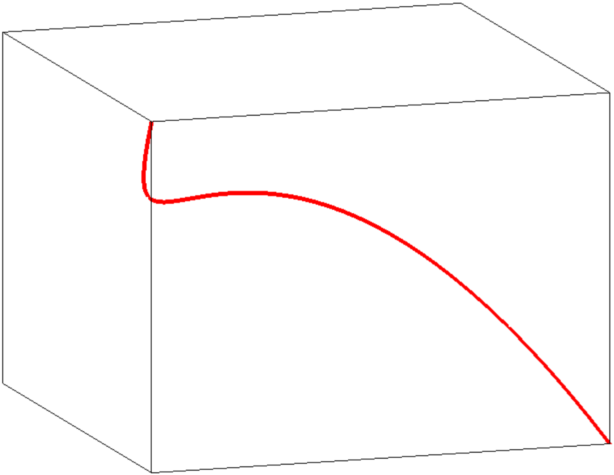

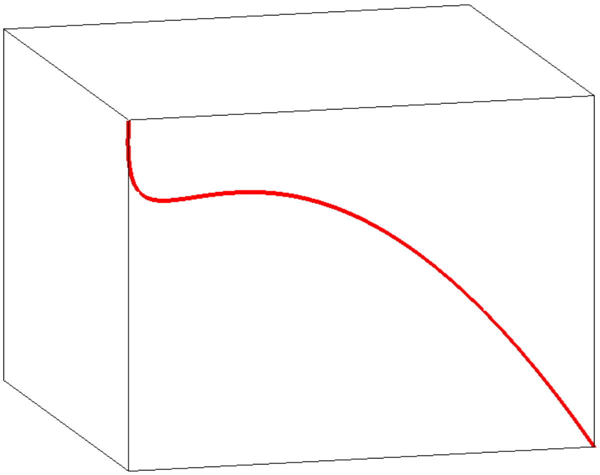

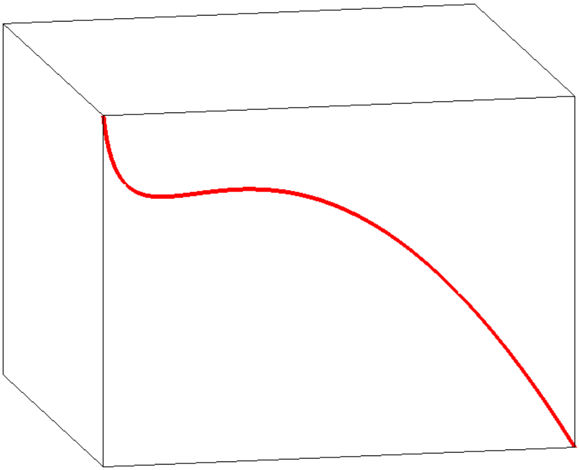

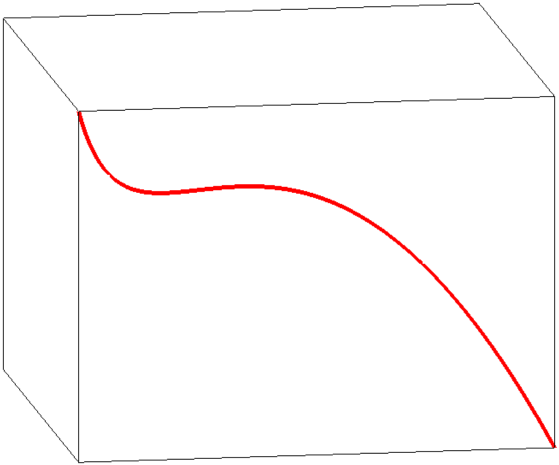

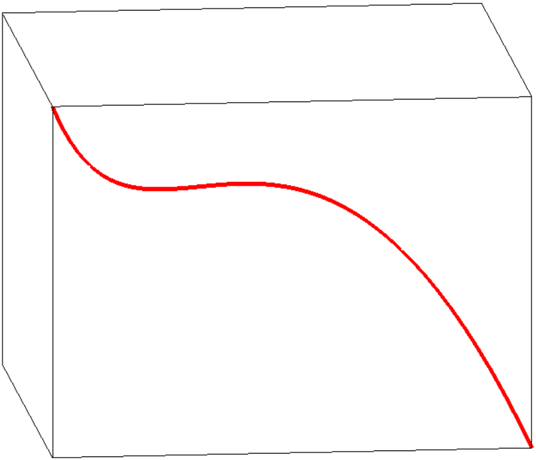

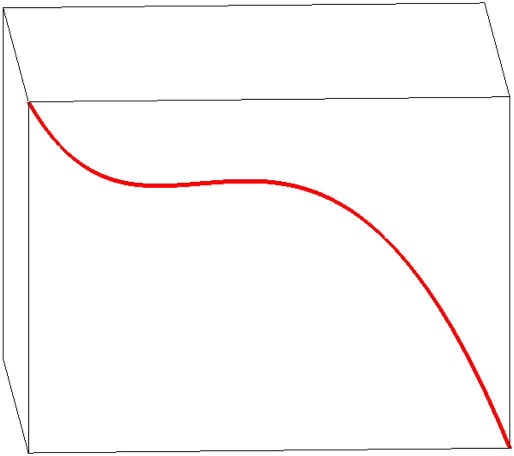

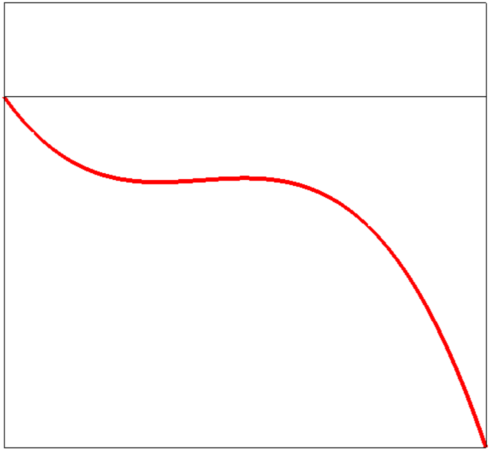

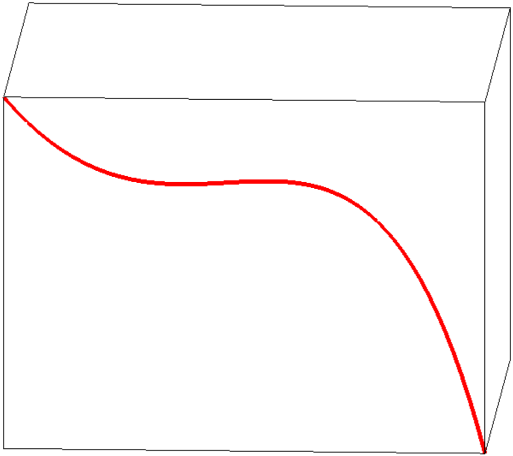

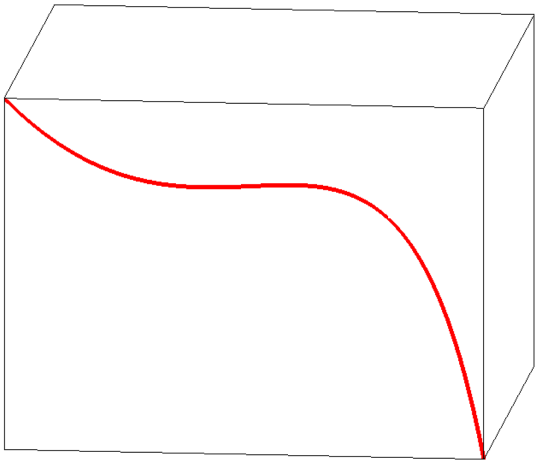

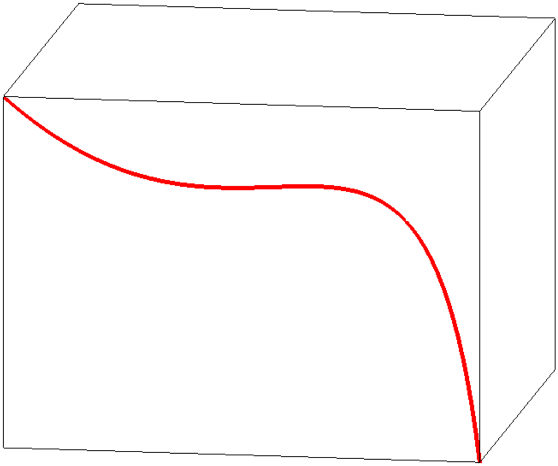

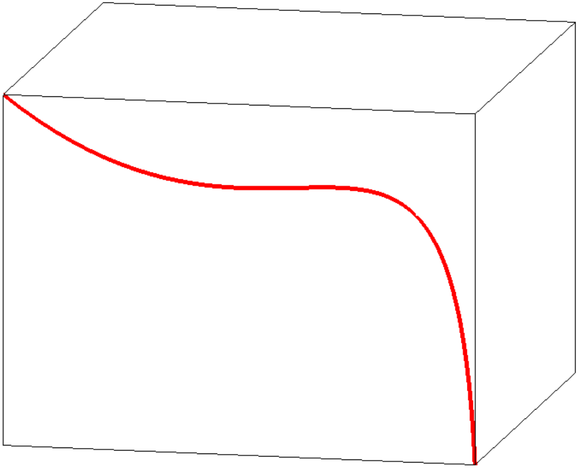

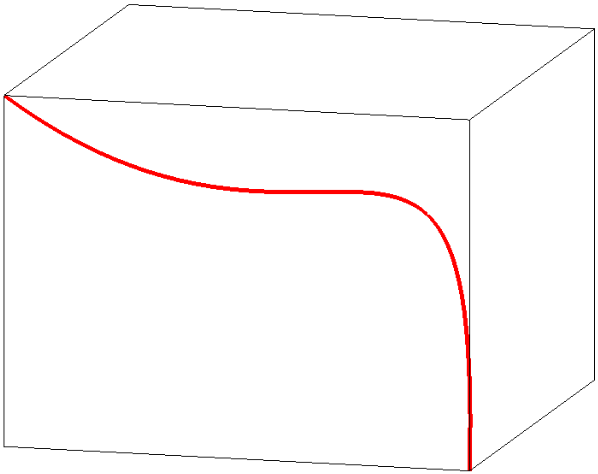

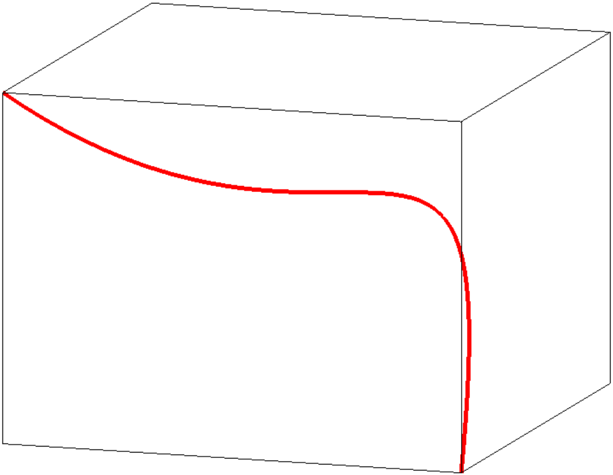

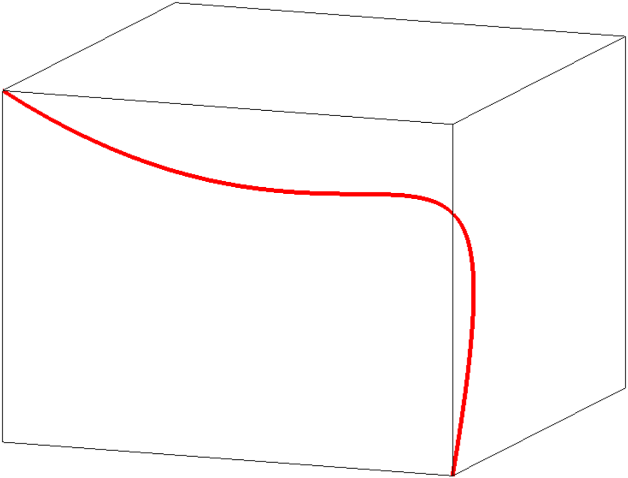

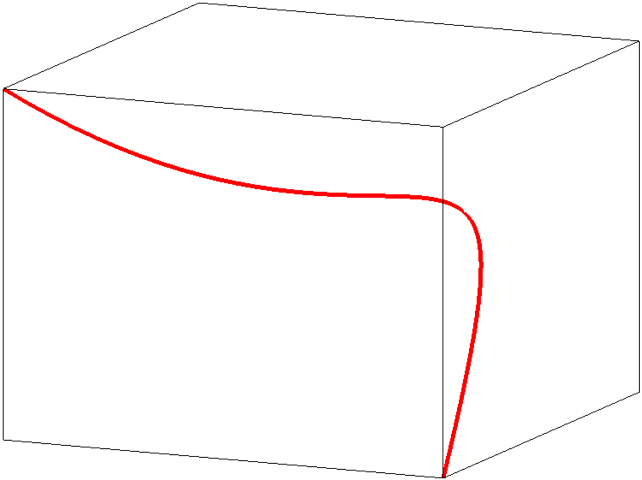

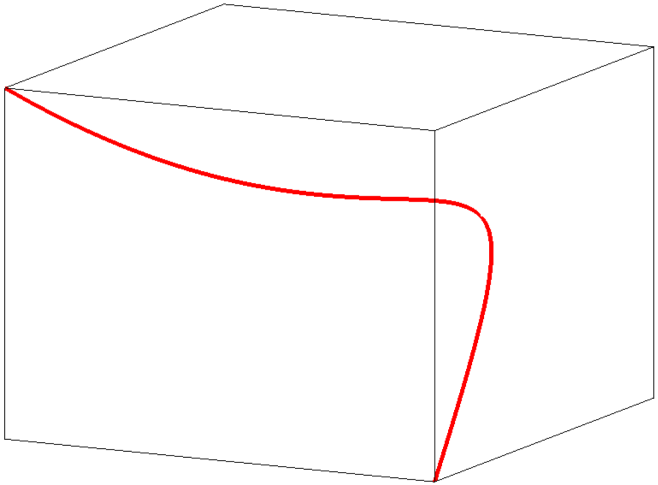

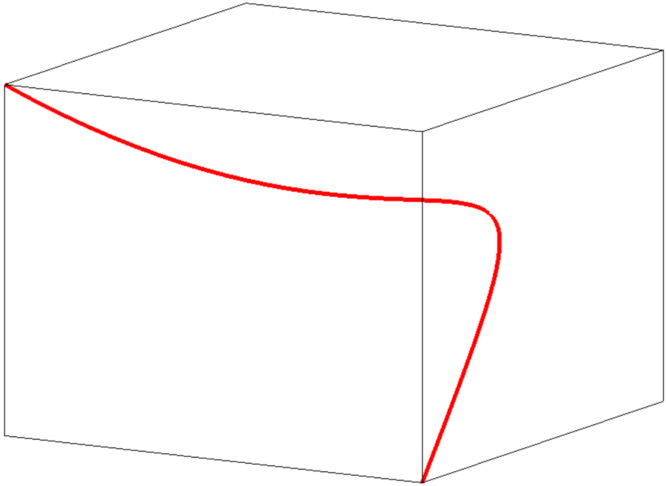

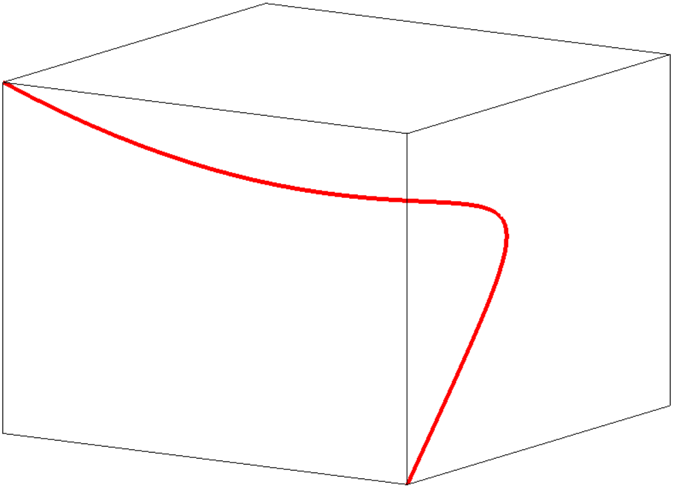

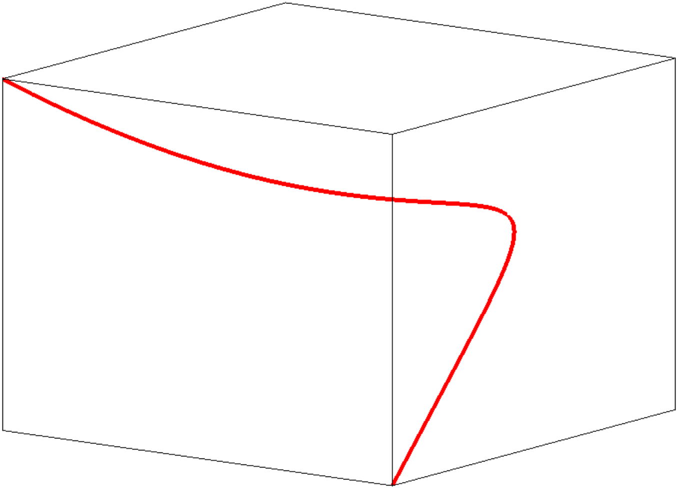

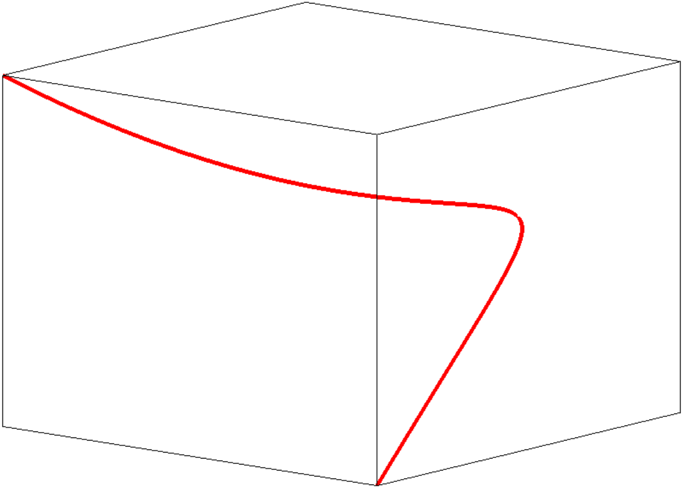

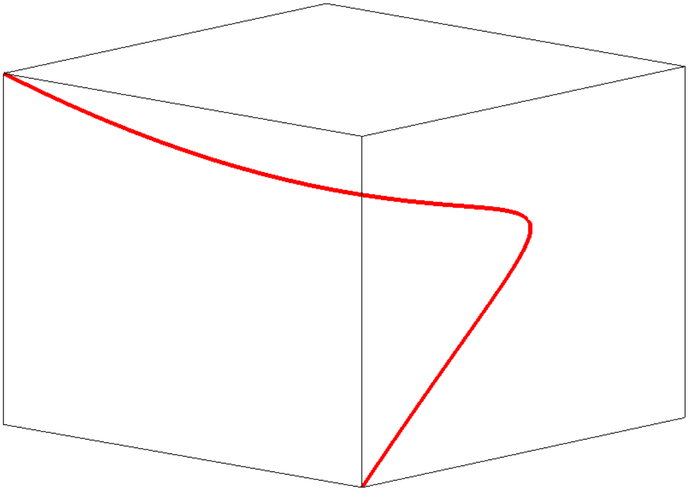

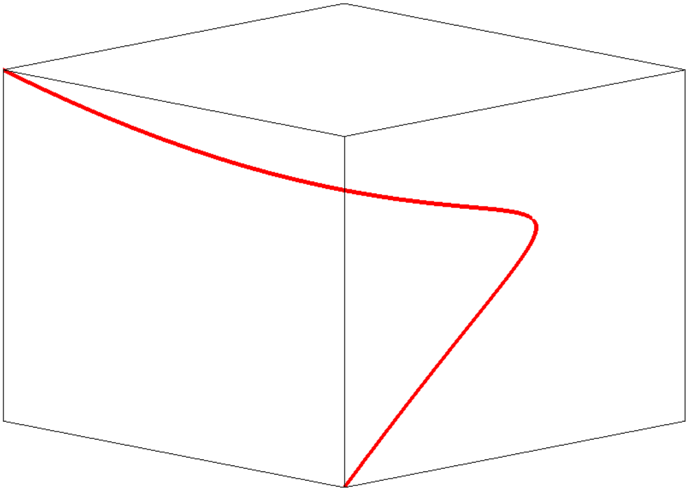

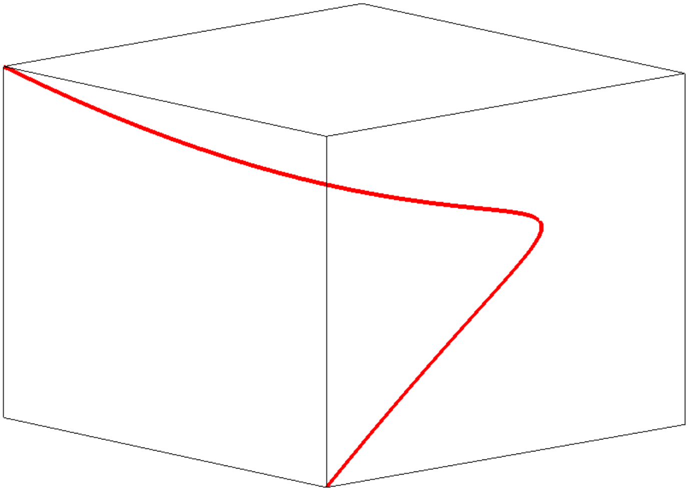

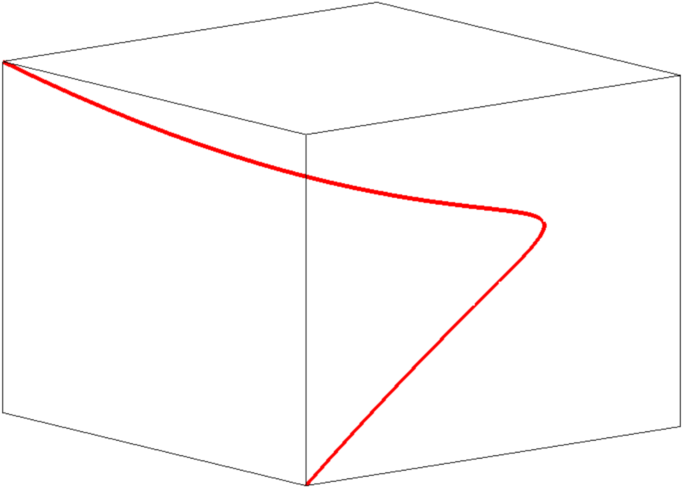

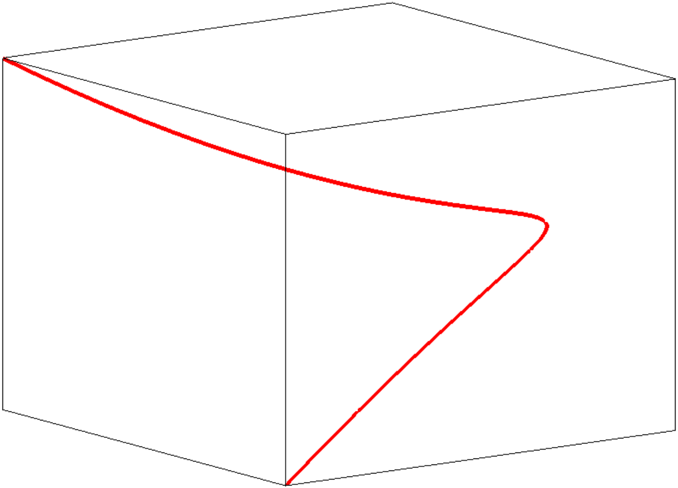

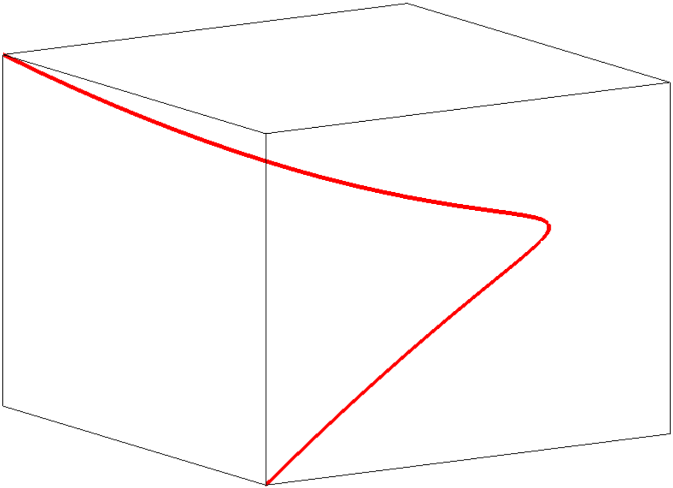

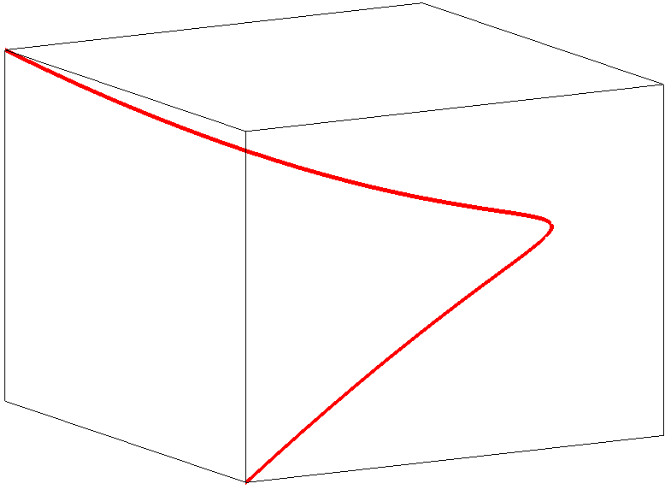

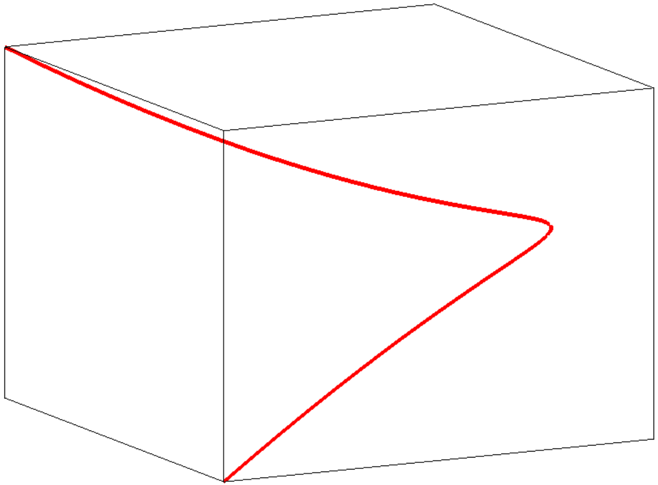

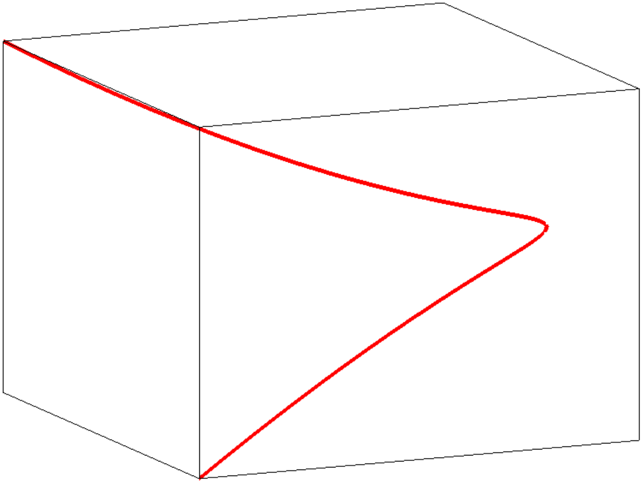

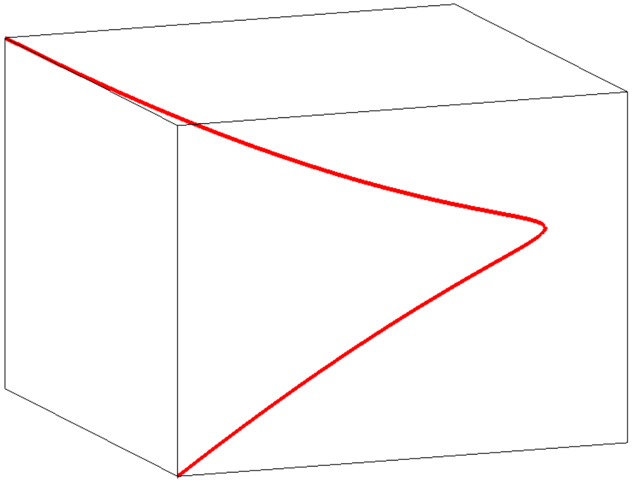

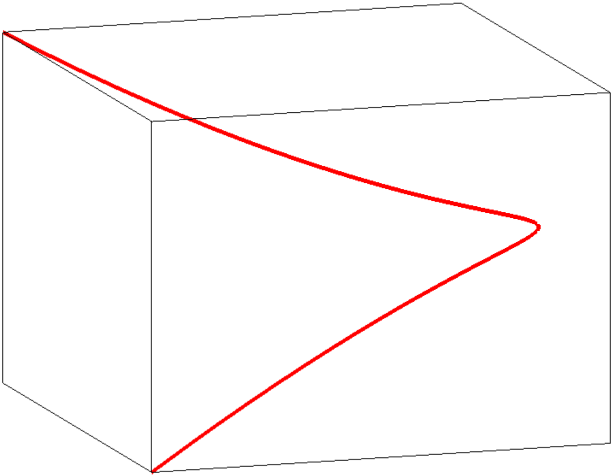

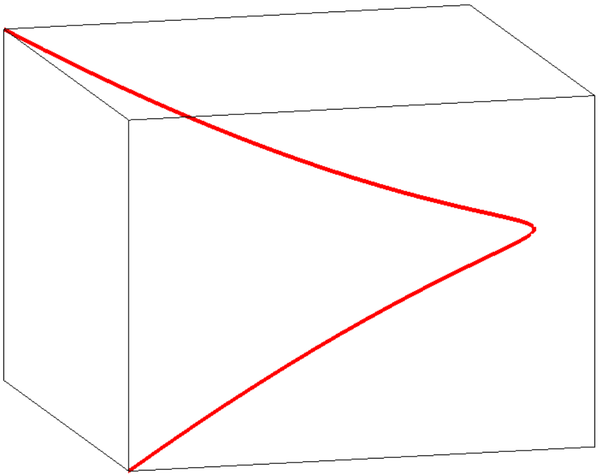

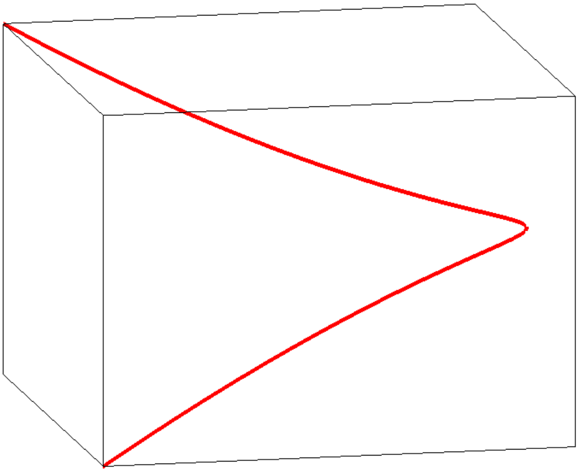

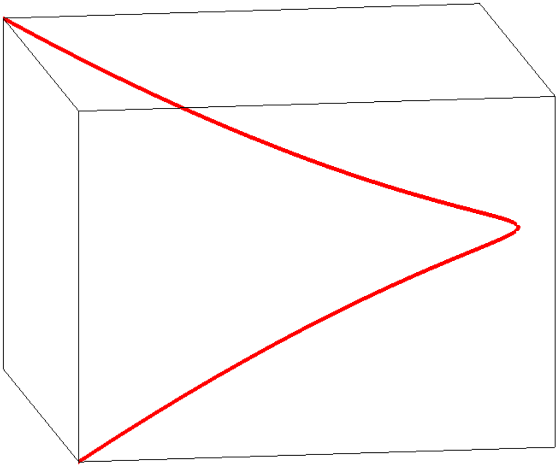

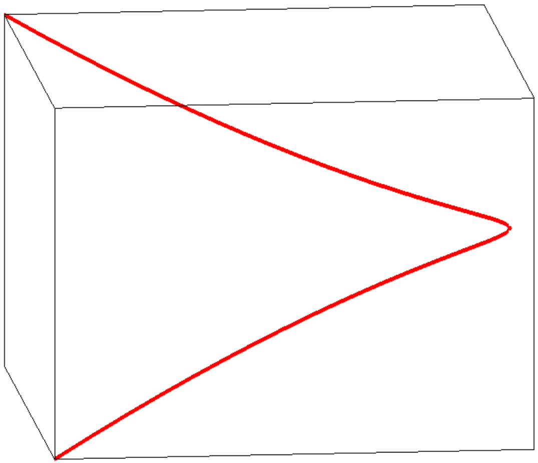

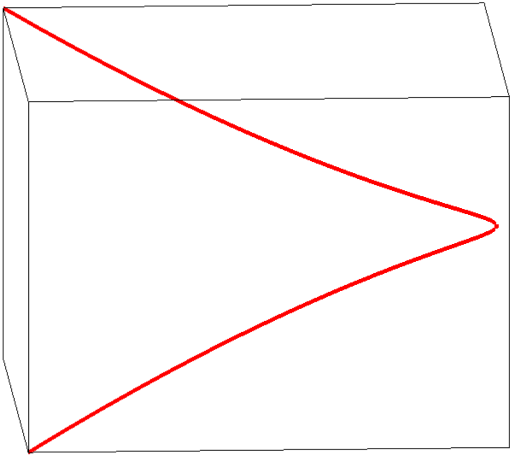

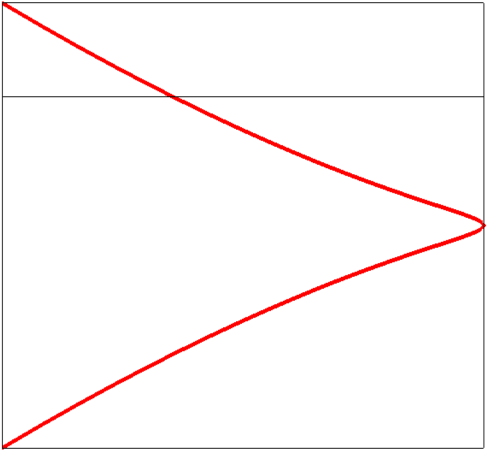

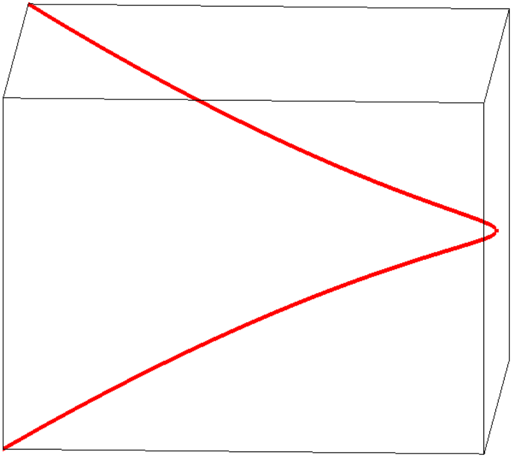

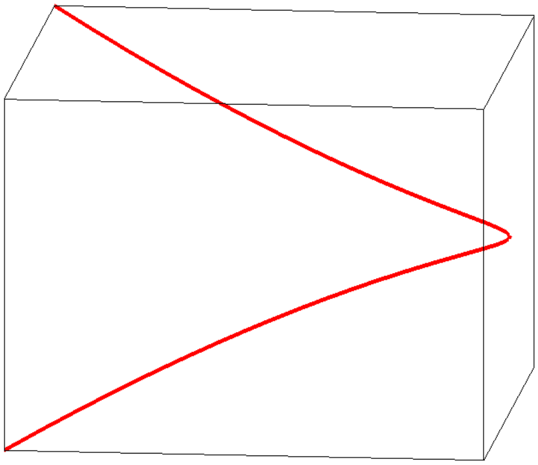

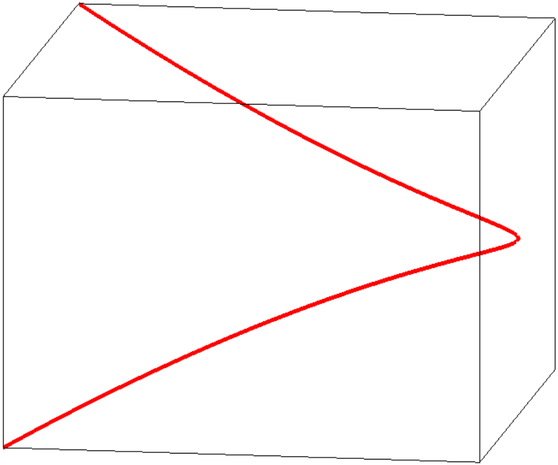

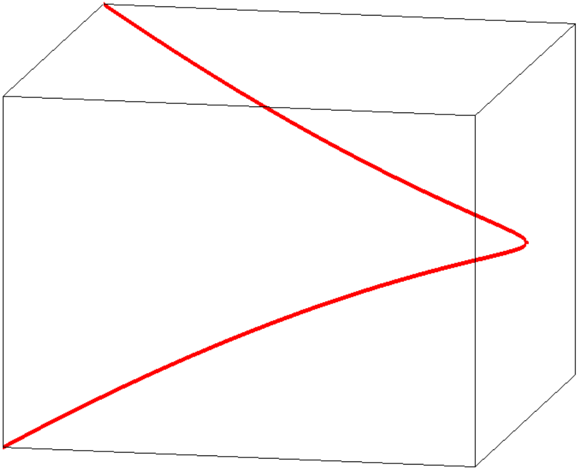

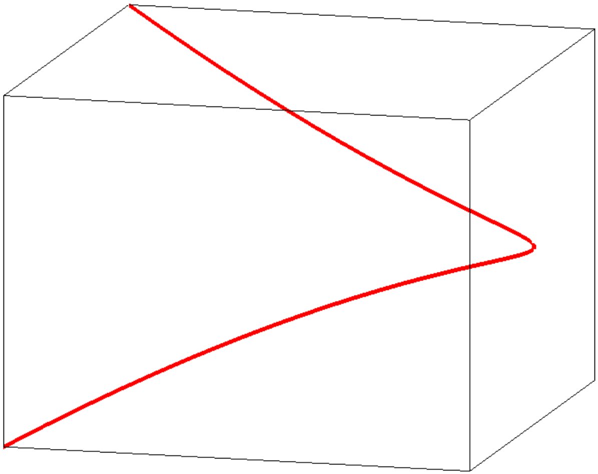

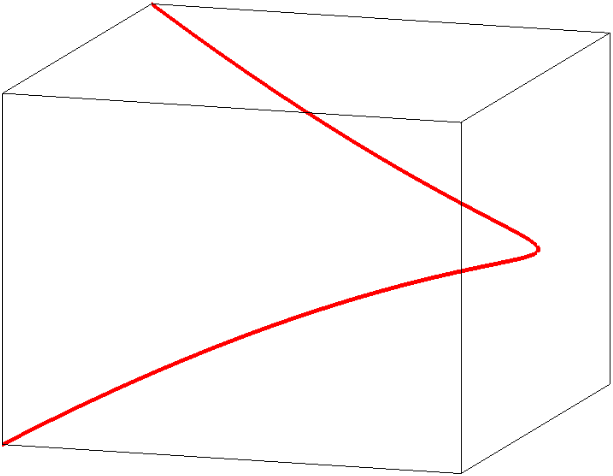

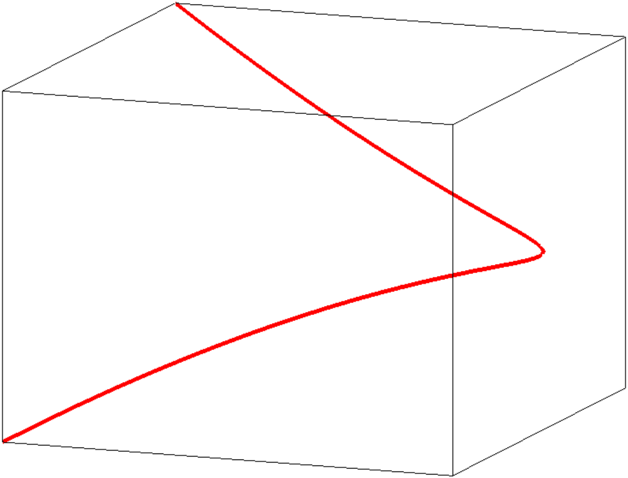

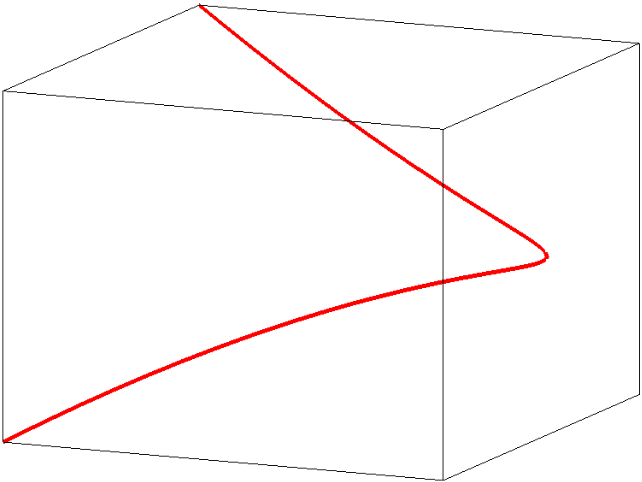

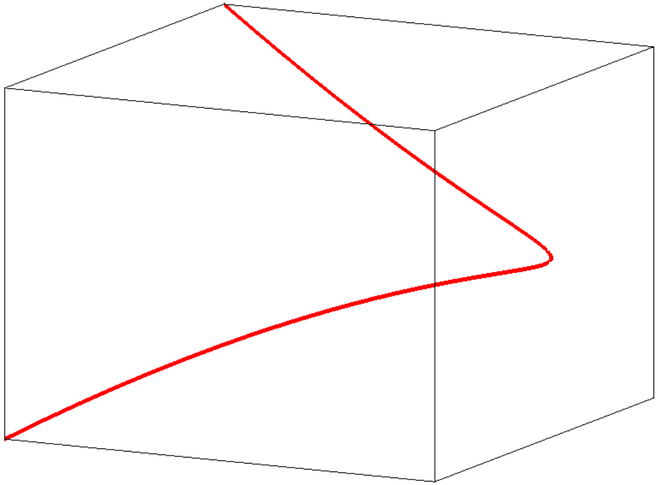

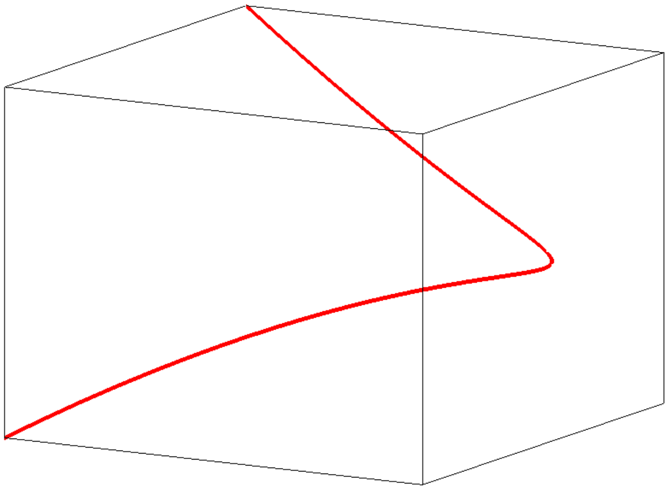

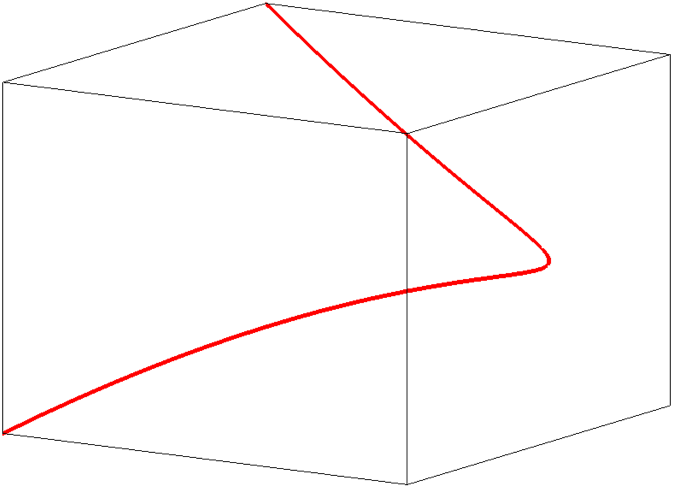

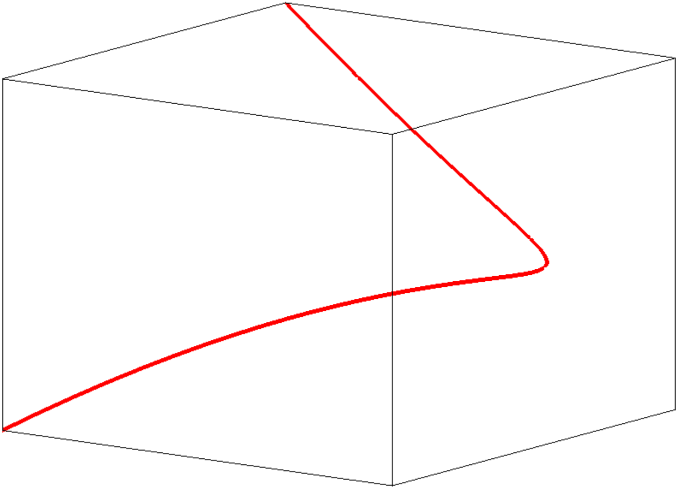

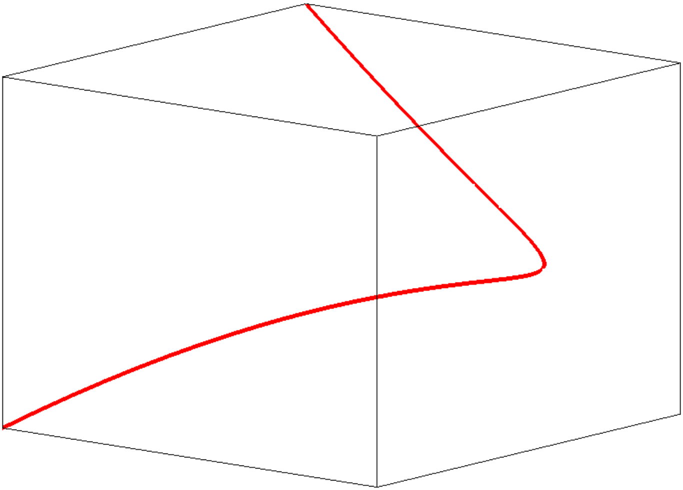

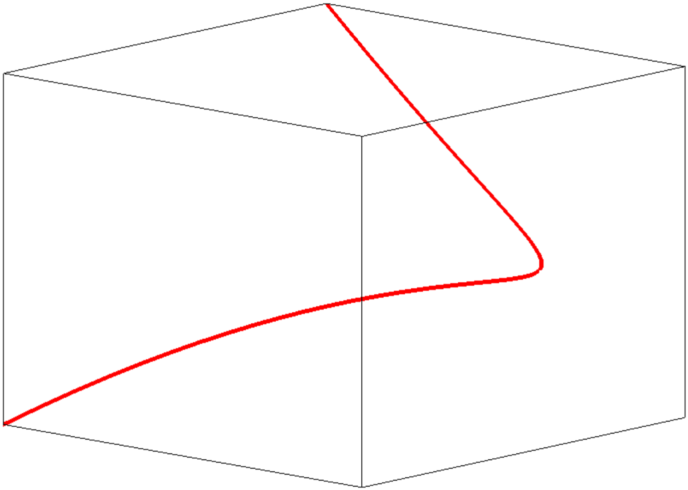

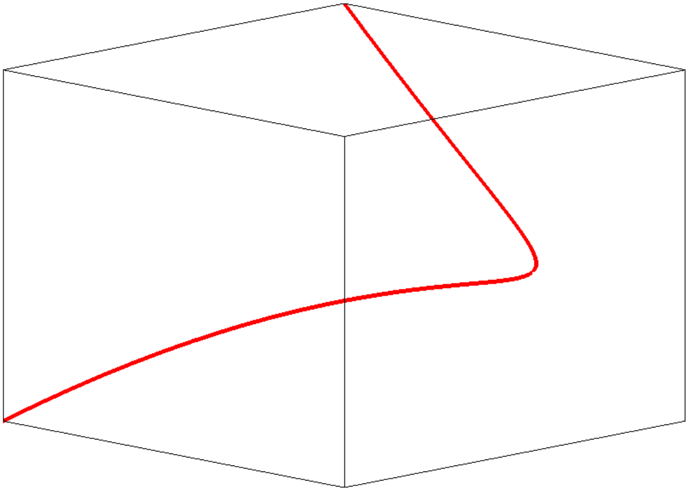

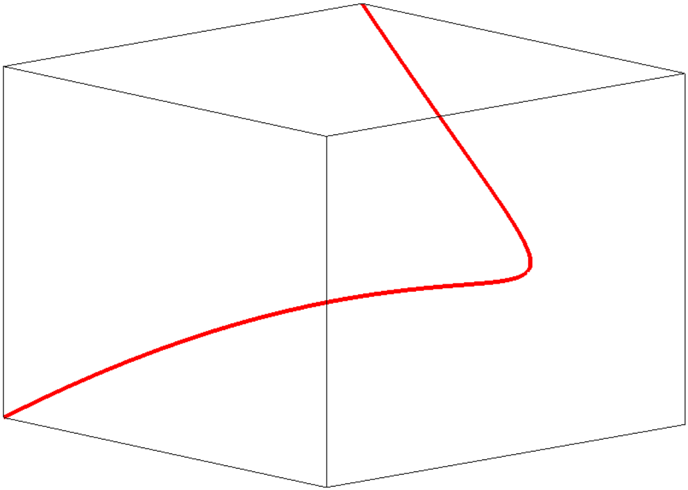

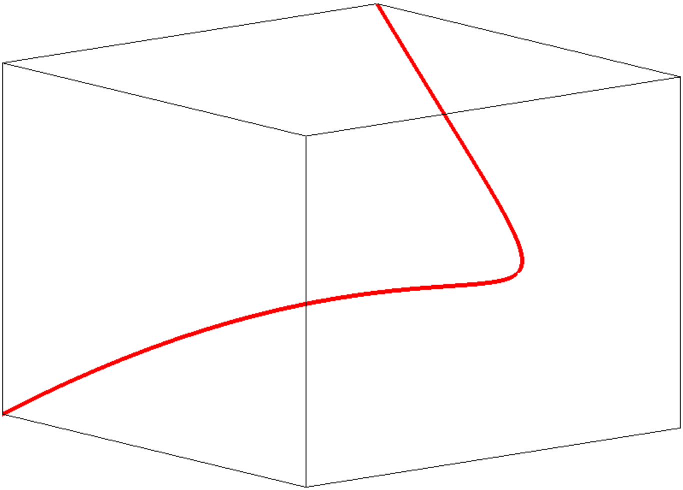

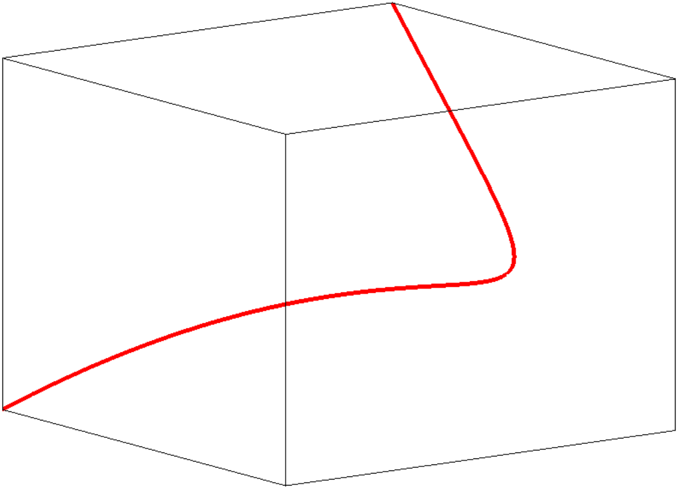

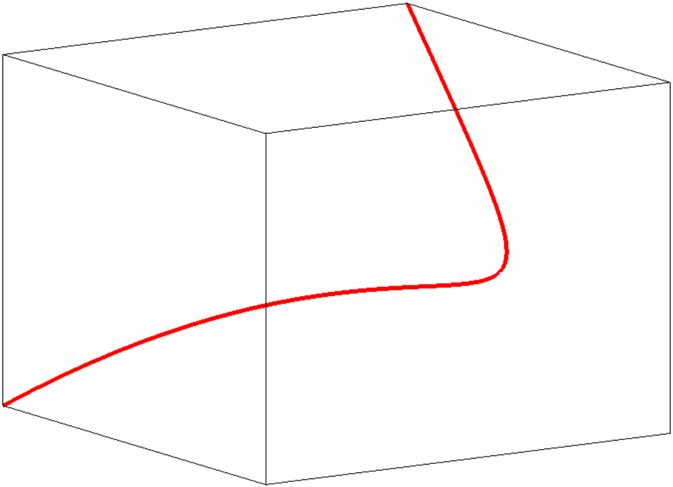

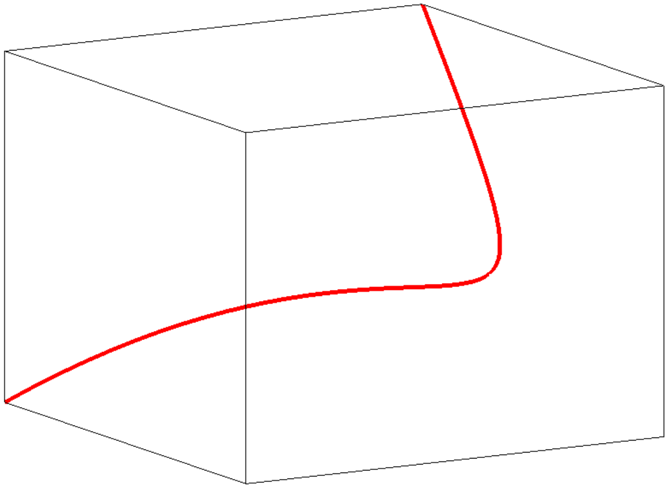

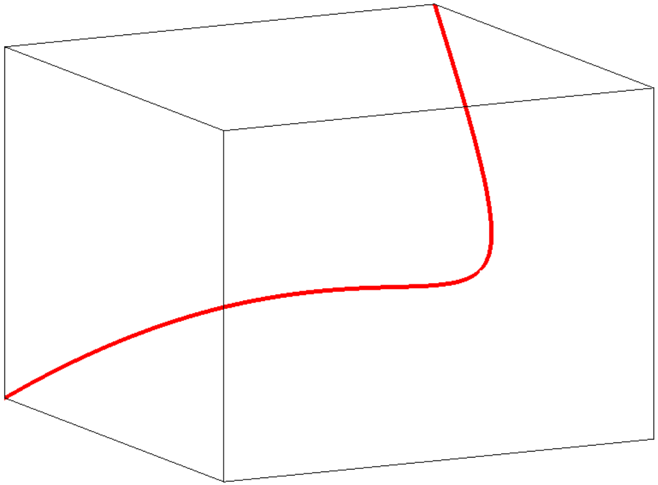

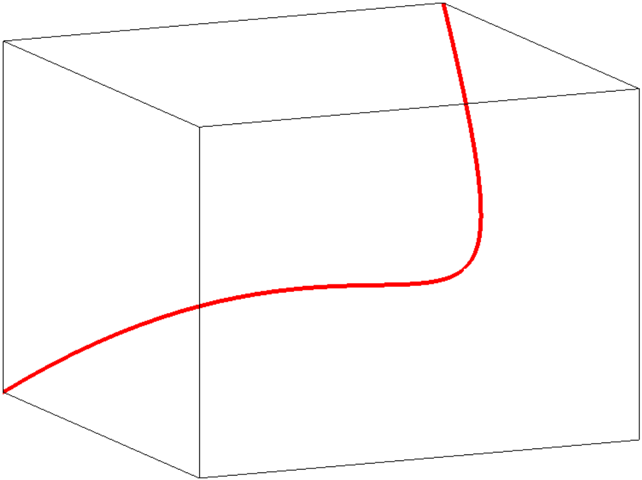

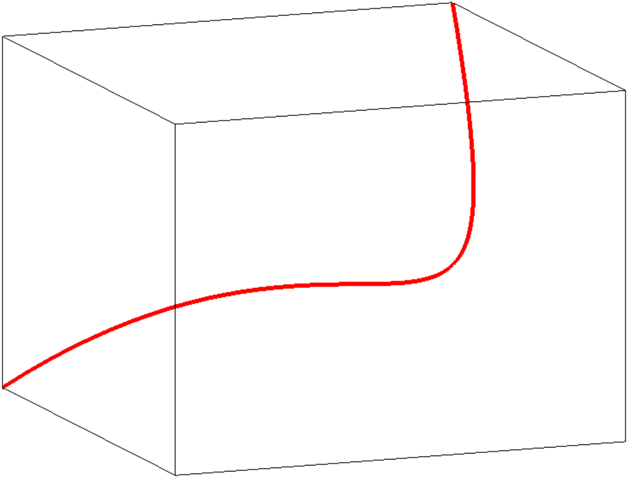

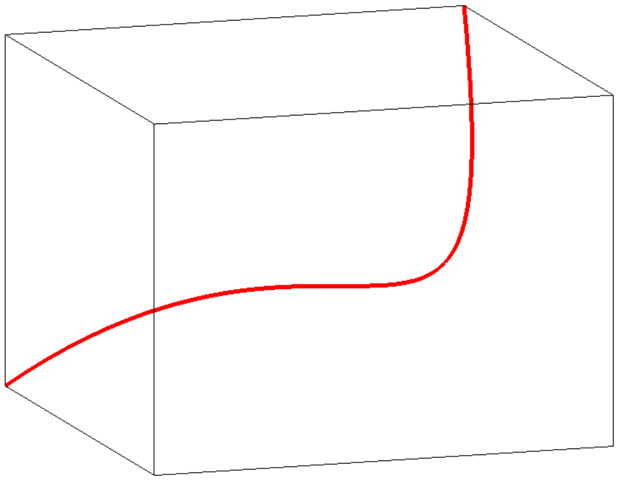

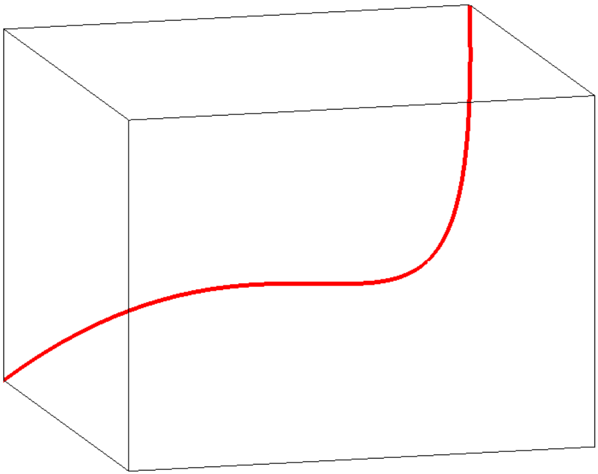

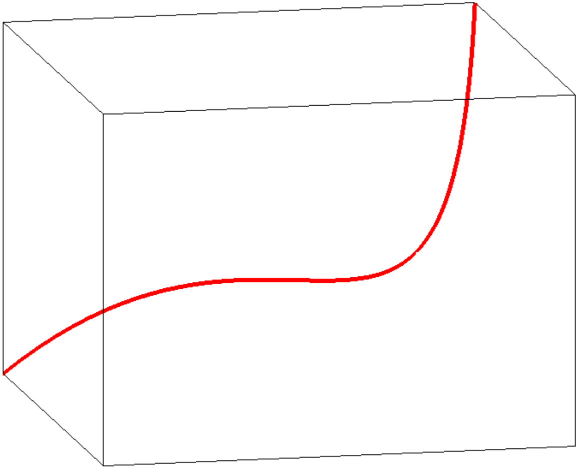

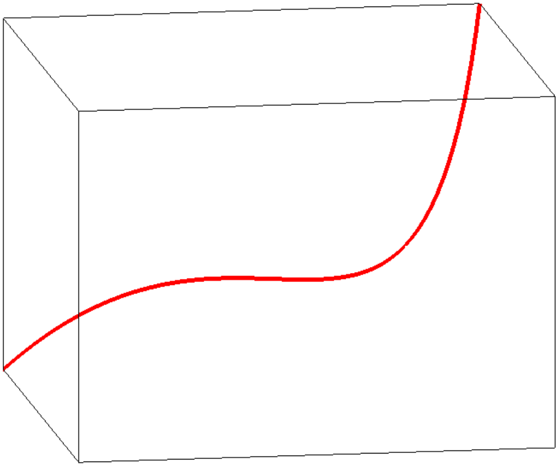

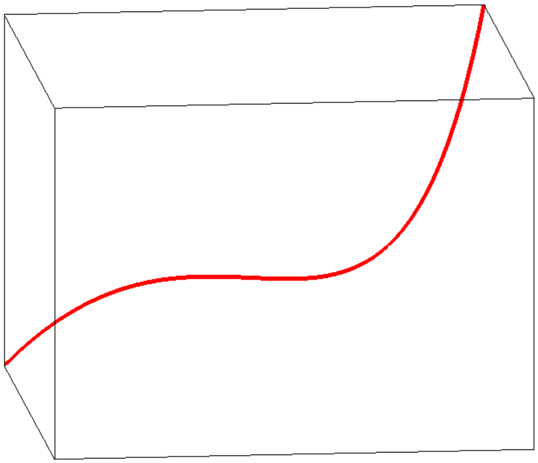

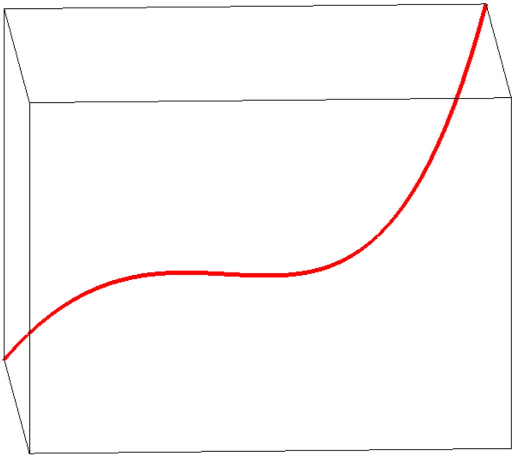

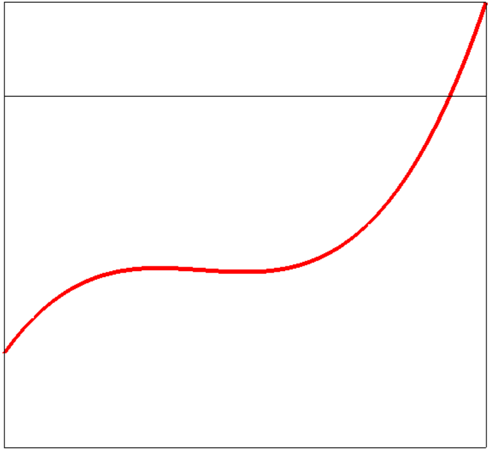

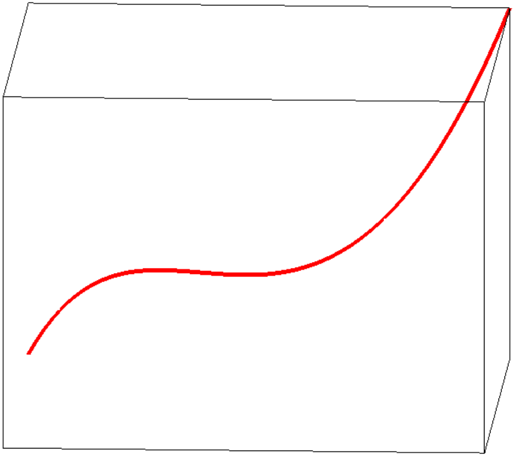

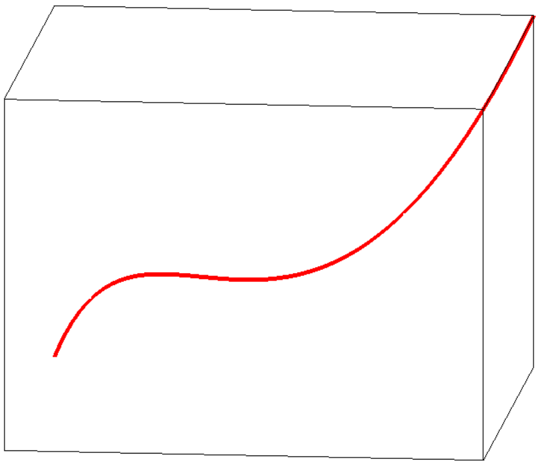

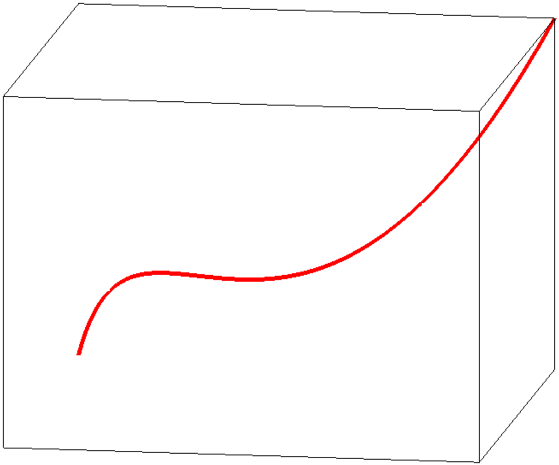

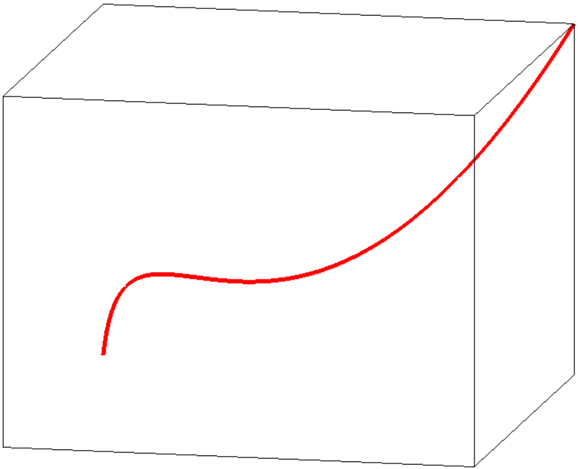

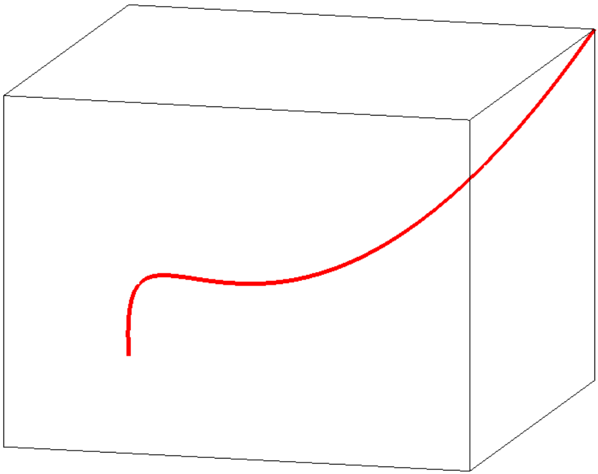

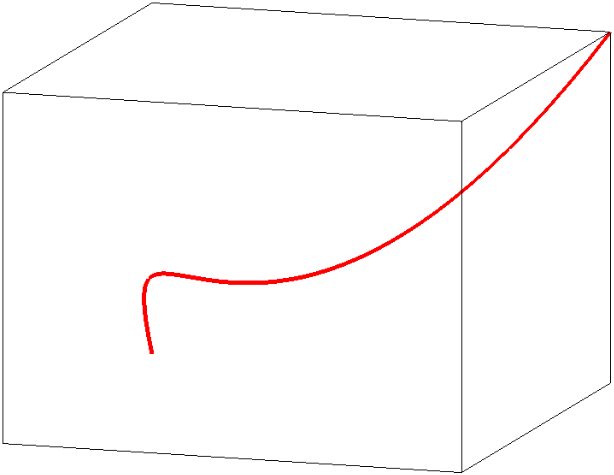

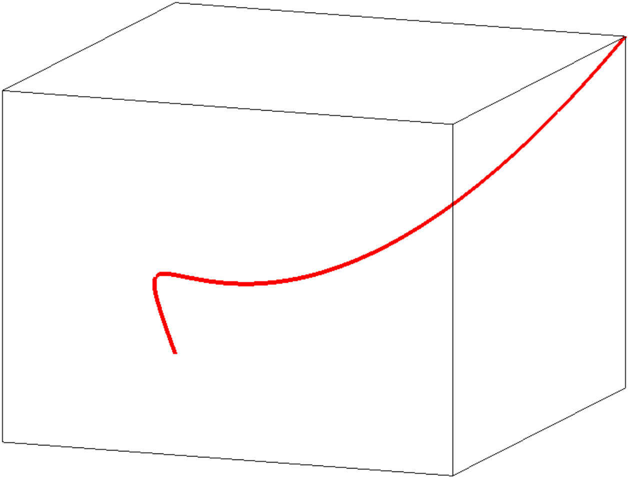

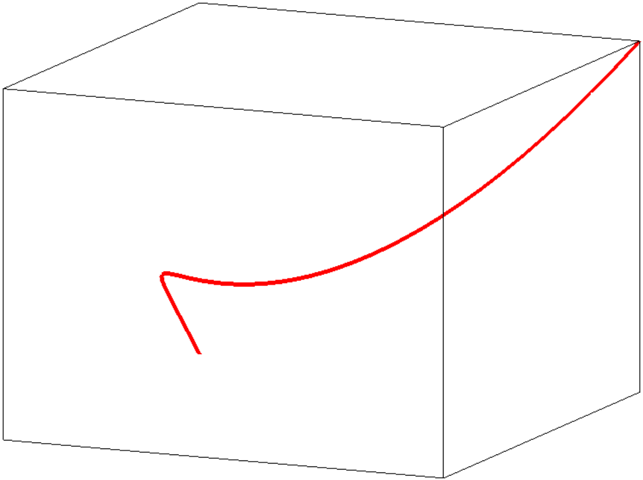

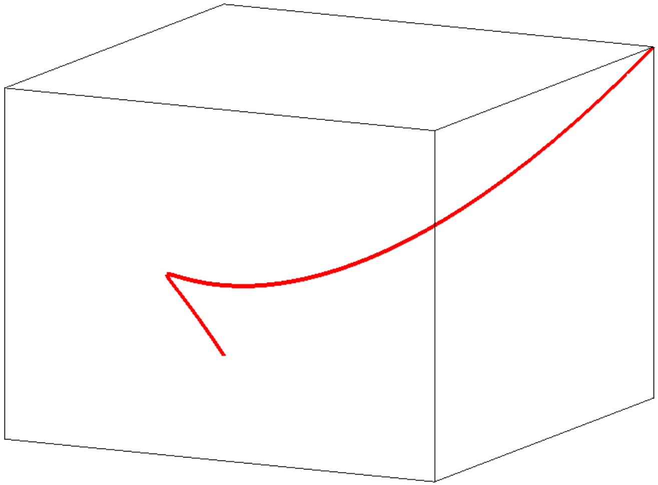

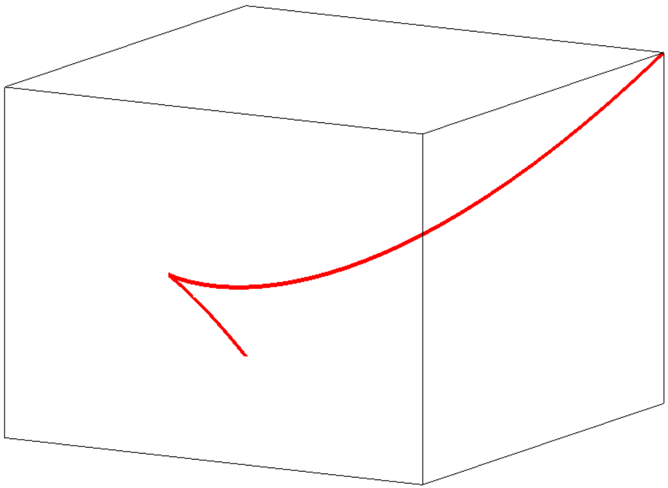

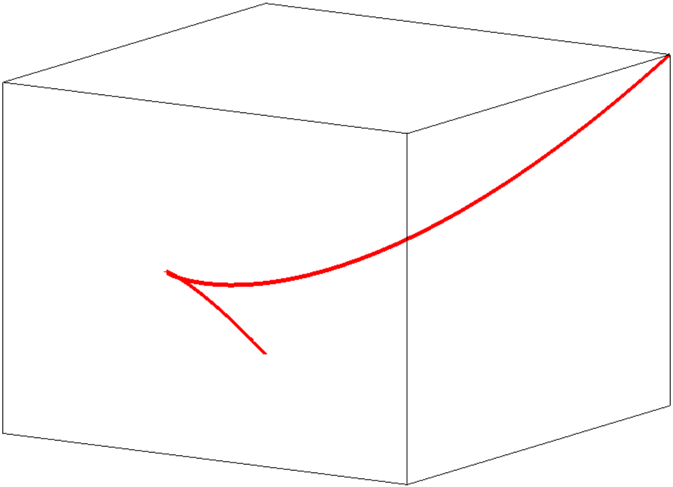

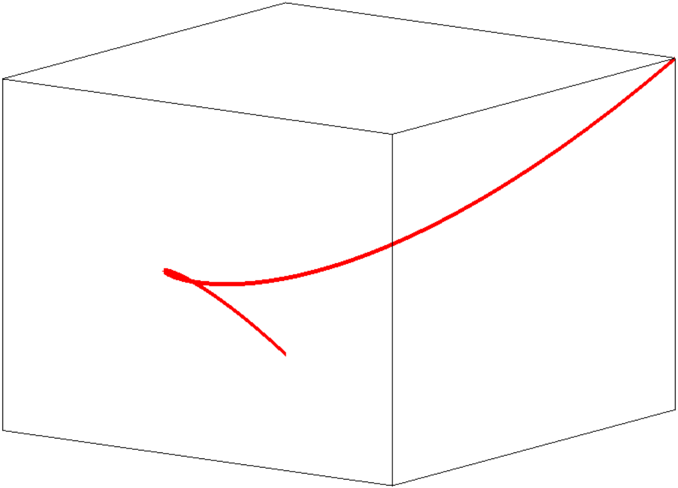

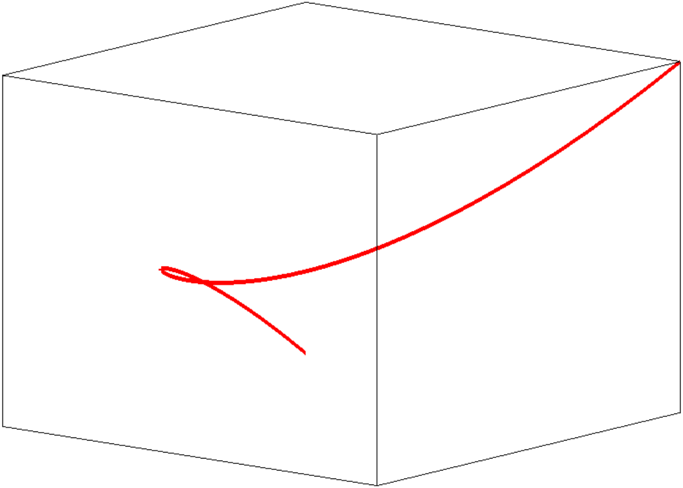

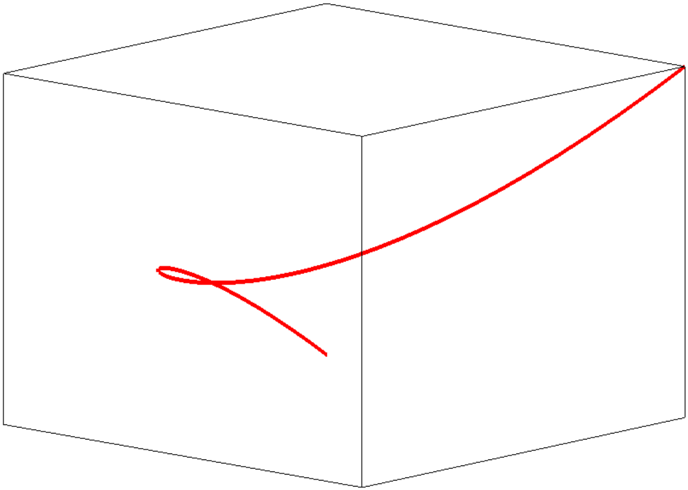

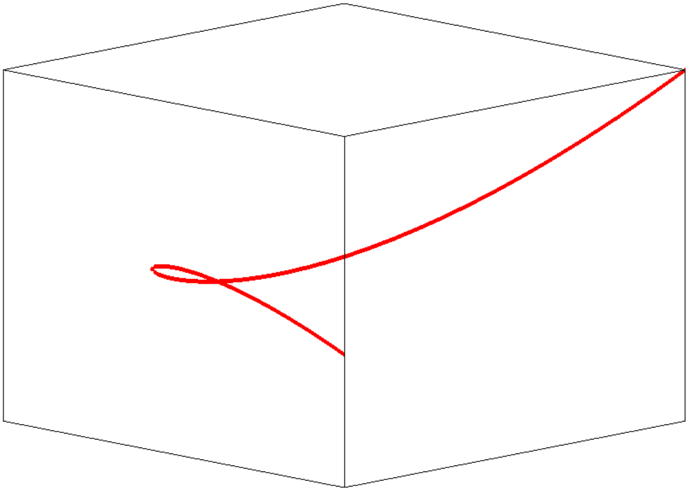

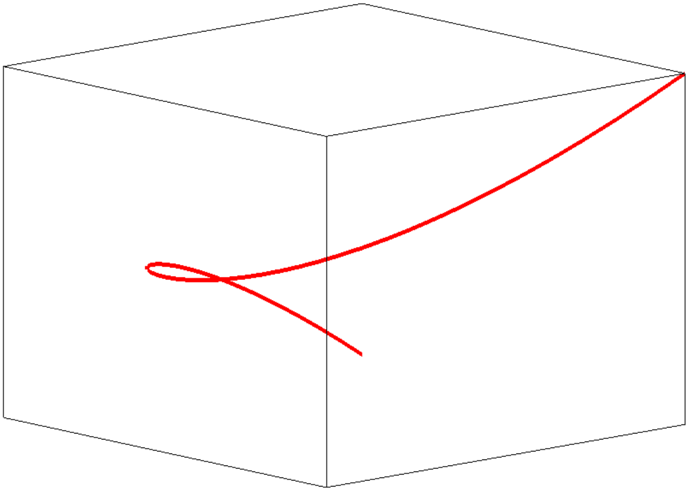

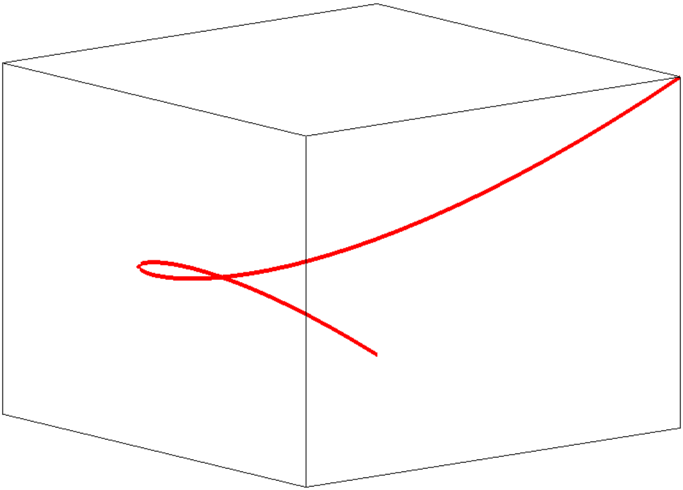

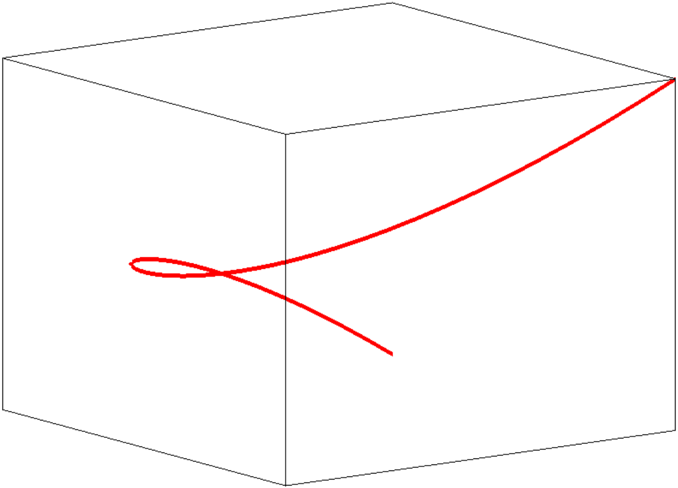

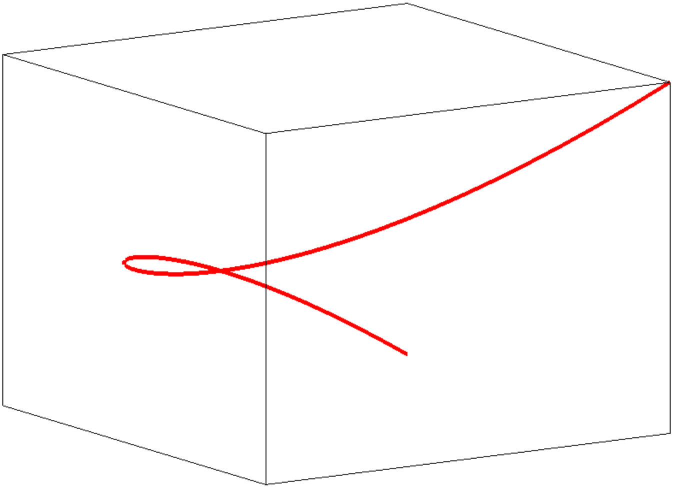

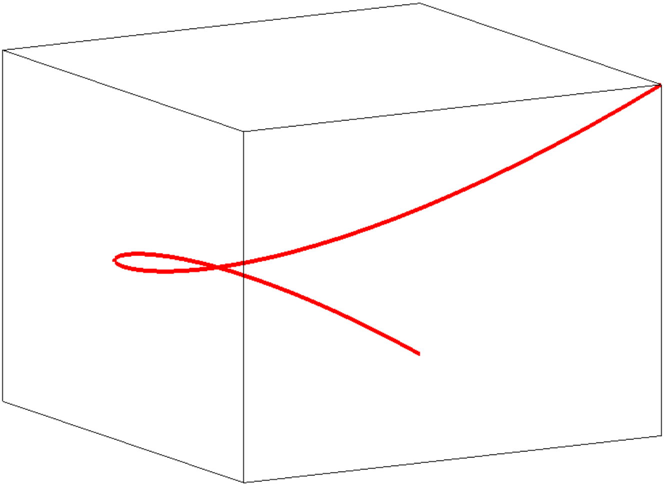

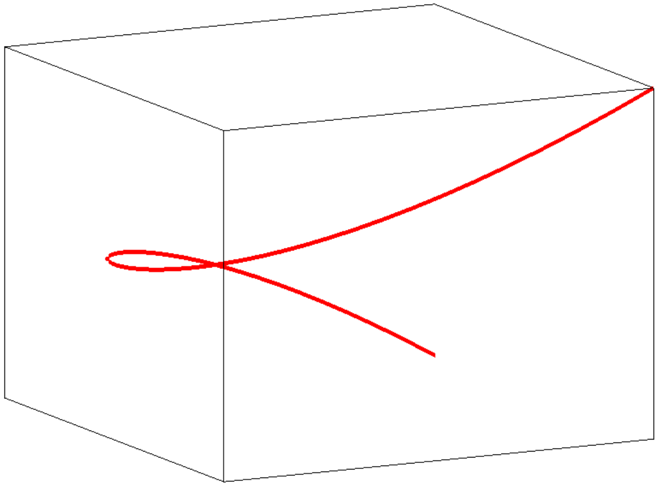

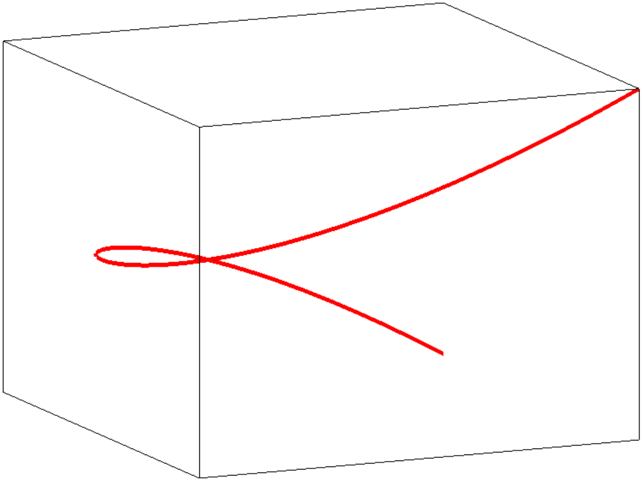

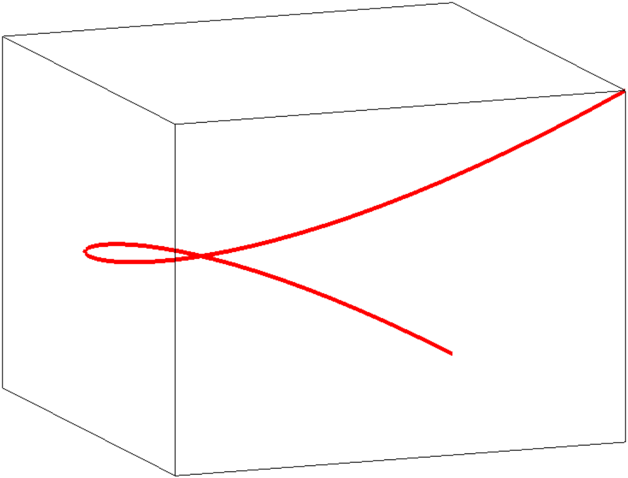

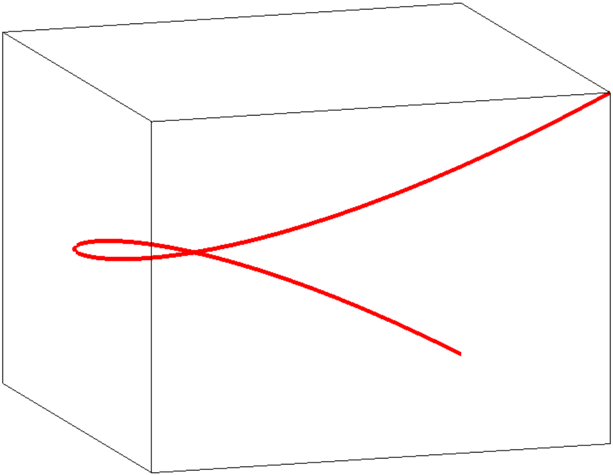

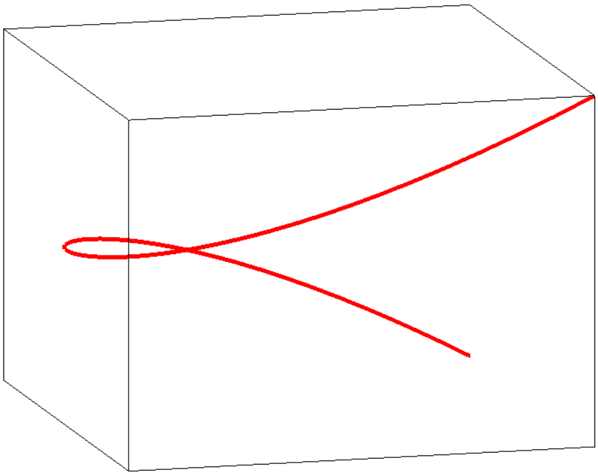

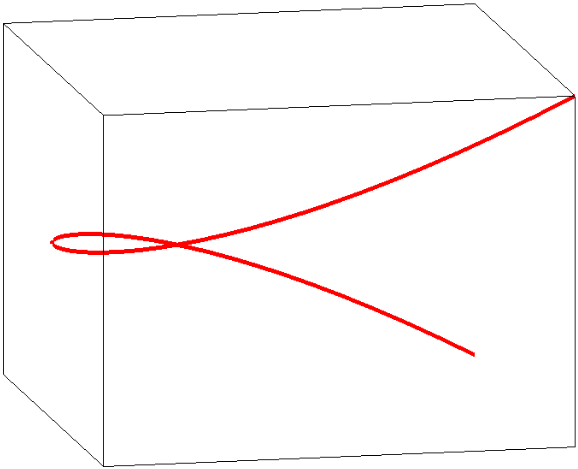

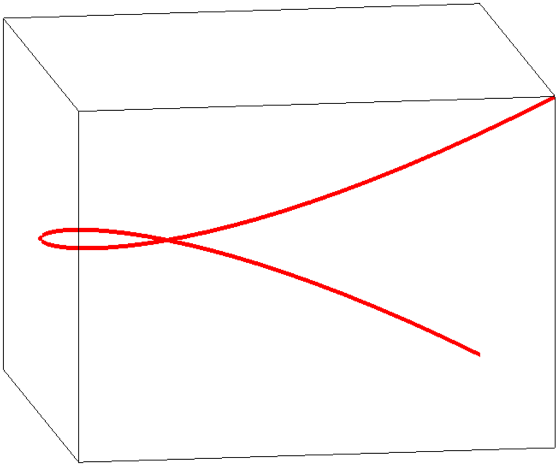

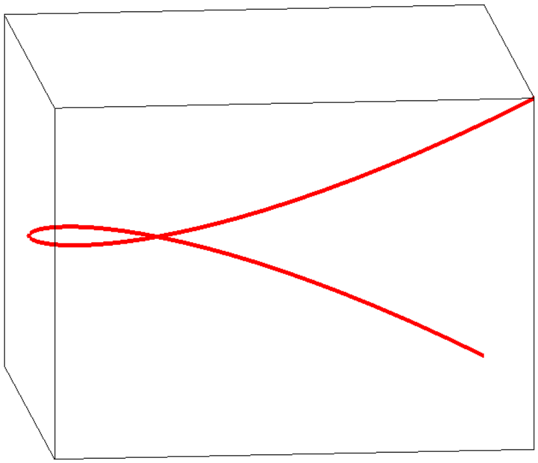

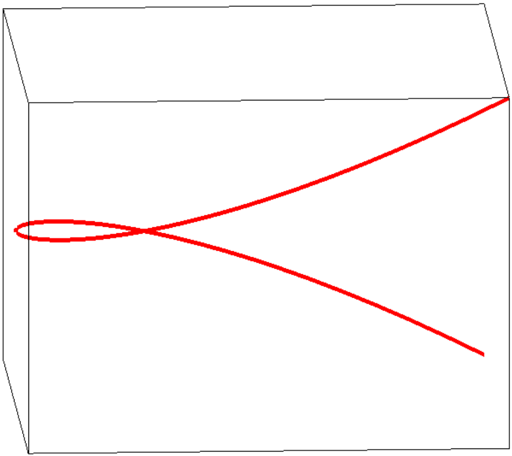

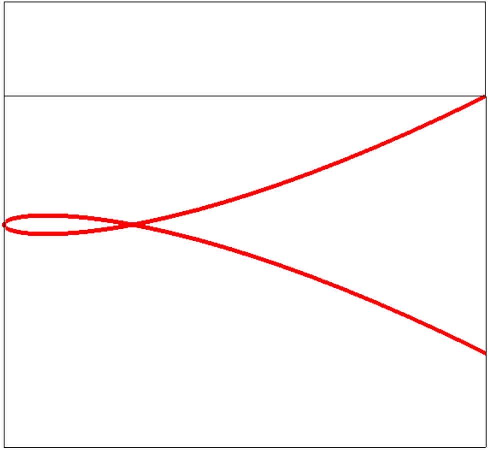

Ideals and Varieties

The affine variety $V(f_1, \ldots, f_r)$ only depends on the ideal

$$I = \langle f_1, \ldots, f_r \rangle \subset R = K[x_1, \ldots, x_n]$$

generated by the f_i: If $f_1(p) = 0, \ldots, f_r(p) = 0$, then

$$\left(\sum_{i=1}^{r} s_i \cdot f_i \right)(p) = \sum_{i=1}^{r} s_i(p) f_i(p) = 0$$

for all $s_i \in R$.

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$$V(I) = \{p \in K^n \mid f(p) = 0 \ \forall f \in I\}$$

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This is indeed an algebraic variety: A ring is called **Noetherian** if every ideal is finitely generated.

Theorem (Hilbert's basis theorem, 1890)

If R is a Noetherian ring, then $R[x]$ is also Noetherian.

Ideals and Varieties

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For a subset $S \subset K^n$ define the **vanishing ideal**

$$I(S) = \{f \in R \mid f(p) = 0 \; \forall p \in S\}$$

Theorem

If K is algebraically closed then

$$\{\text{prime ideals of } R\} \quad \overset{V}{\underset{I}{\rightleftarrows}} \quad \{\text{irreducible affine varieties in } K^n\}$$

is a bijection.

Projection and Elimination

Gaussian elimination parametrizes the solution set L of a linear system of equations. It computes a bijective projection $L \to K^r$. In the case of non-linear systems we can proceed in a similar way:

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How to compute I_m?

Division with Remainder

For an ideal $I = \langle f_1, \ldots, f_r \rangle$ in the Euclidean domain (and hence PID) $K[x]$, the Euclidean algorithm computes a generator of $I = \langle \mathrm{ggT}\,(f_1, \ldots, f_r) \rangle$. Using division with remainder (successively removing the term of highest degree), we can test whether $f \in K[x]$ is in I.

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Using division with remainder (successively removing the term of highest degree), we can test whether $f \in K[x]$ is in I.
For $f \in R = K[x_1, ..., x_n]$ one has to choose a **lead term** $\mathrm{LT}(f)$, e.g., by ordering the terms **lexicographically** w.r.t $x_1 > ... > x_n$.

Example

We divide $x^2y + xy^2 + y^2$ by $xy - 1$ and $y^2 - 1$ for $x > y$:

$$
\begin{array}{l}
x^2y + xy^2 + y^2 = \quad x\,(xy-1) + y\,(xy-1) + x + 1\,(y^2-1) + y + 1 \\
x^2y - x \\
\hline
xy^2 + x + y^2 \\
xy^2 - y \\
\hline
x + y^2 + y \\
\hline
y^2 + y \\
y^2 - 1 \\
\hline
y + 1
\end{array}
$$

We divide $x^2 - y^2$ by $x^2 + y$ and $xy + x$

$$\begin{array}{rl} x^2 - y^2 = & 1 \cdot (x^2 + y) + (-y^2 - y) \\ x^2 + y & \\ \hline -y^2 - y & \end{array}$$

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This is strange! We have

$$x^2 - y^2 = -y\left(x^2 + y\right) + x\left(xy + x\right)$$

hence $x^2 - y^2 \in I = \left\langle x^2 + y, xy + x \right\rangle$, but the remainder is not zero.

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Gröbner basis in the example: $G = \left\{y^2 + y, x^2 + y, xy + x\right\}$

Gröbner Bases and Ideal Membership in Singular

Theorem

If G is a Gröbner basis of $I \subset R$ and $f \in R$, then $f \in I$ iff $\mathrm{NF}(f, G) = 0$.

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Gröbner Bases and Projection

Theorem

If $G = \{g_1, \dots, g_r\}$ *is a lexicographical Gröbner basis of* $I \subset K[x_1, \dots, x_n]$, *then*

$$G_m = G \cap K[x_{m+1}, \dots, x_n]$$

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Using projections, we can compute the image of $V(I)$ under a rational map $\varphi = (\varphi_1, \dots, \varphi_r)$ with $\varphi_i \in K(x_1, \dots, x_n)$ by projecting the graph

$$\begin{array}{ccccc} & & \Gamma(\varphi) & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ V(I) & & \dashrightarrow & & K^r \end{array} \qquad = \{(x, \varphi(x)) \mid x \in V(I)\} \subset K^n \times K^r$$

Desingularization of Curves

Given a singular curve, find smooth curve C' which is birational to C.

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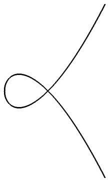

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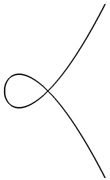

The **coordinate ring** of C (i.e. ring of functions on C) is the quotient ring

$$A = K[x, y]/I \quad \text{where} \quad I = \left\langle x^3 + x^2 - y^2 \right\rangle .$$

Suppose C' is another affine variety with coordinate ring A'.

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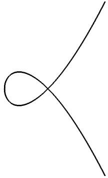

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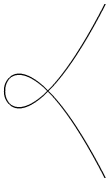

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Equivalently: There is a rational map $C \dashrightarrow C'$ defined on a Zariski open subset of C which admits an inverse rational map.

Desingularization of Curves

Example

$$C = V(x^3 + x^2 - y^2) \xrightarrow{\varphi} K^1,\ (x,y) \longmapsto t = \frac{y}{x}$$

$$I(\Gamma(\varphi)) = \langle x^3 + x^2 - y^2,\ x \cdot t - y,\ \overbrace{w \cdot x - 1}^{\Rightarrow x \neq 0} \rangle \subset K[w,x,y,t]$$

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Elimination with $w > x > y > t$:

$x^3 + x^2 - y^2$		w		
$xt - y$	$-w$		wy	$-wt$
$wx - 1$	t	$x^2 - x$		1
$-wy + t$	1		xt	t
$-wy^2 + x^2 + x$		1	-1	
$-x^2 + xt^2 - x$			1	w
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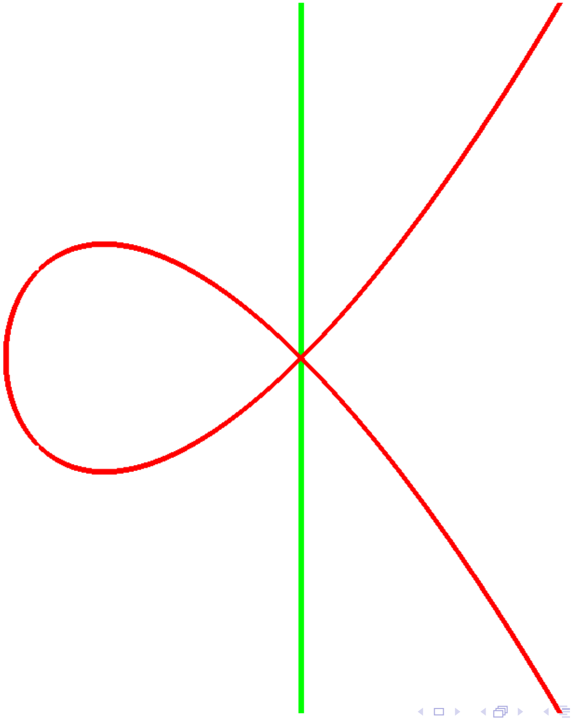
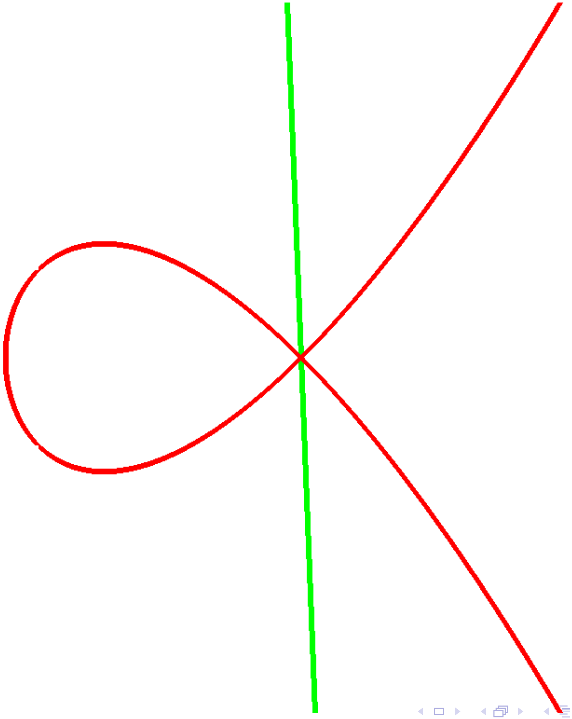
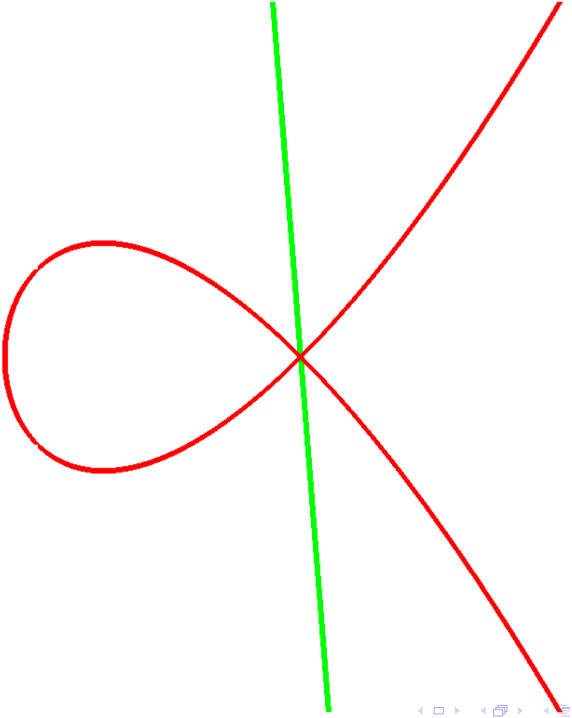
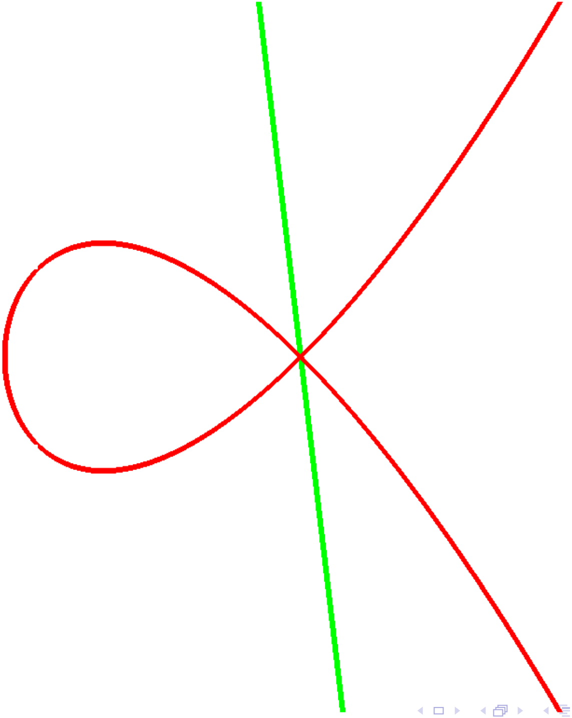
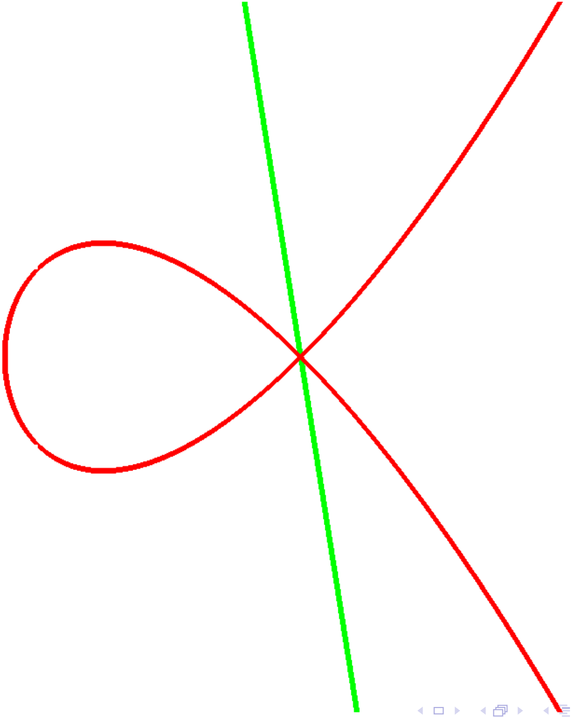
$$K^1 \xrightarrow{\varphi^{-1}} C$$
$$t \longmapsto (t^2 - 1, t^3 - t)$$

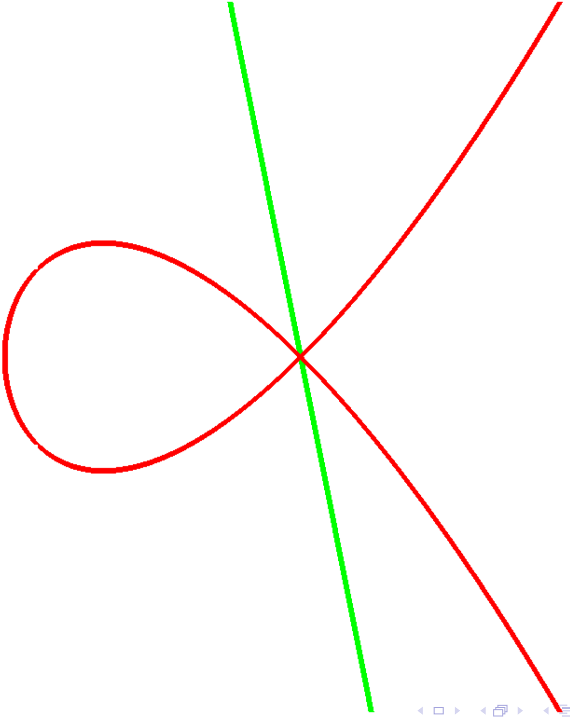


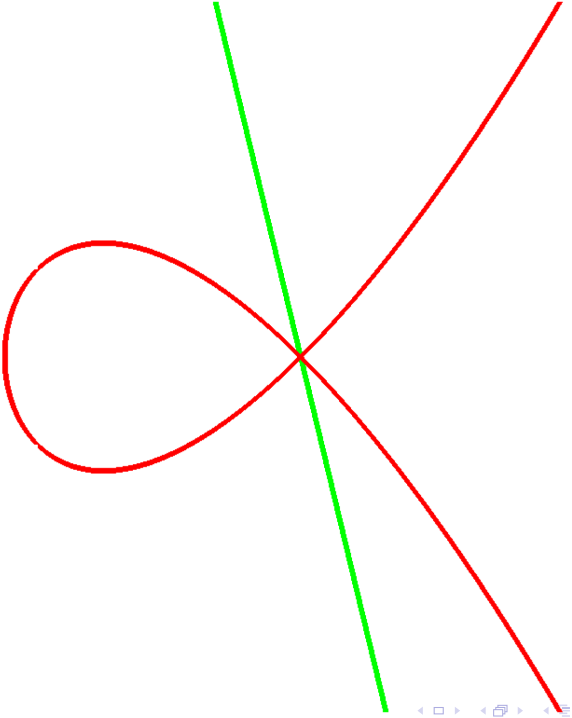


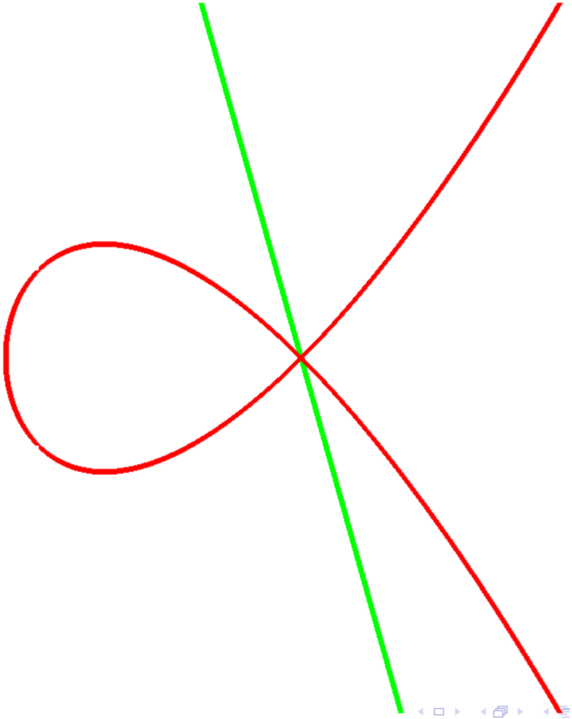


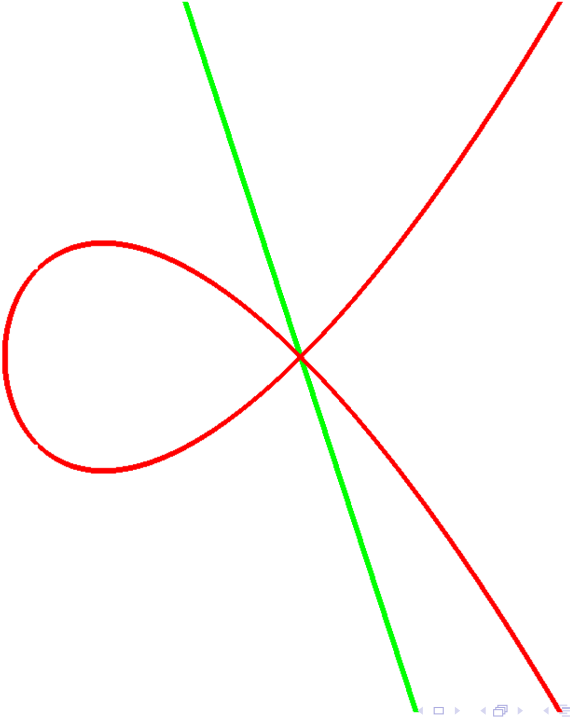


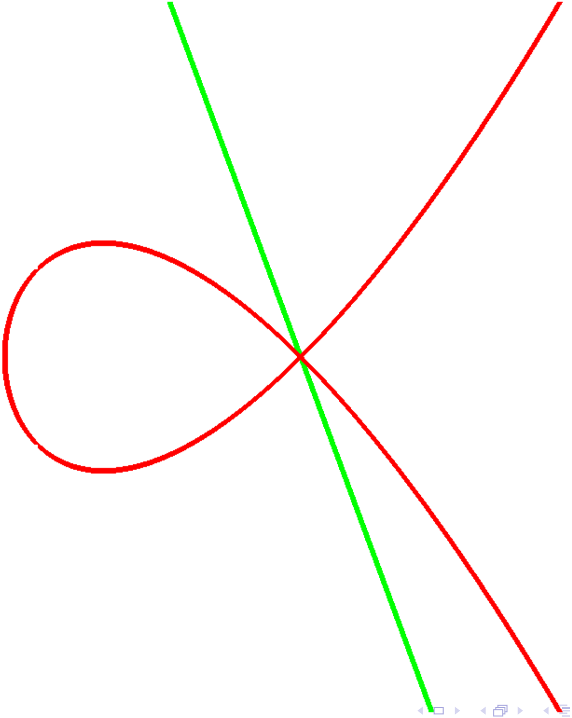


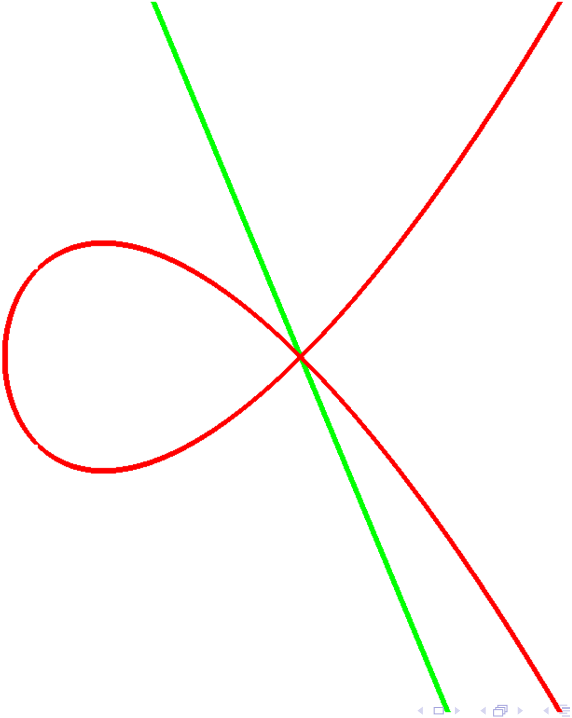


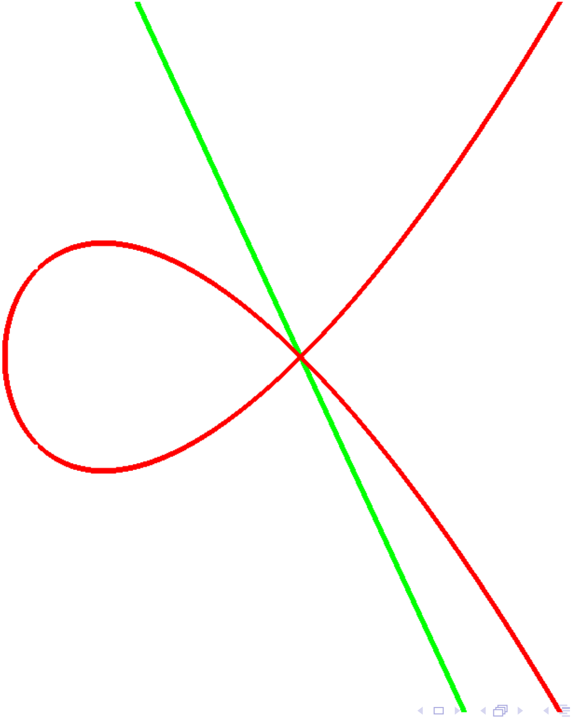


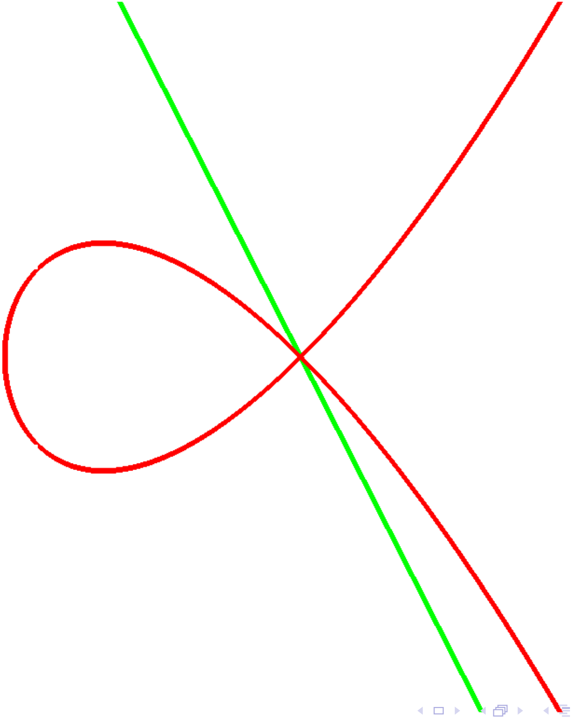


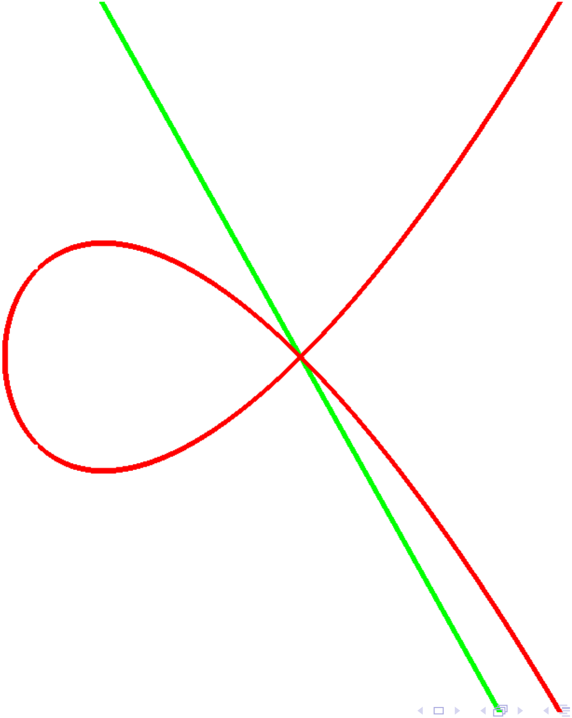


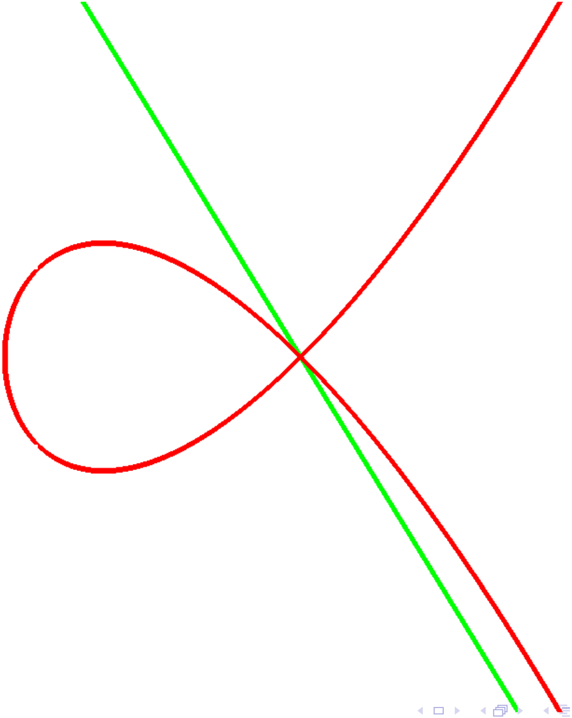


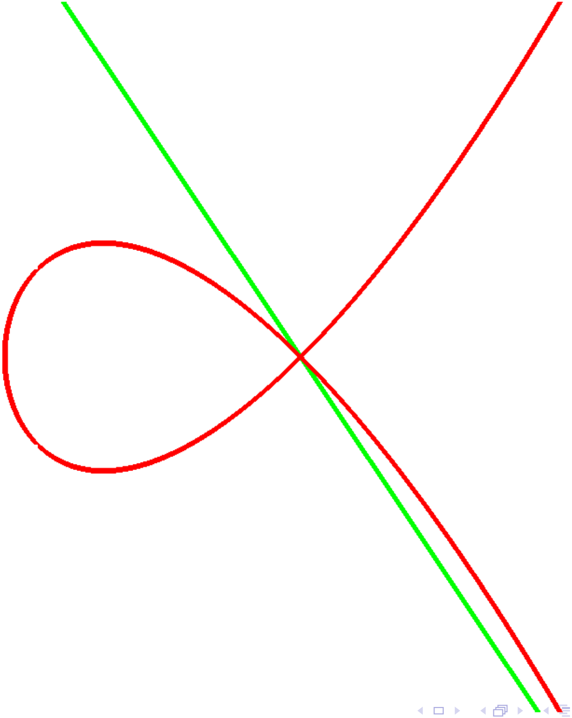


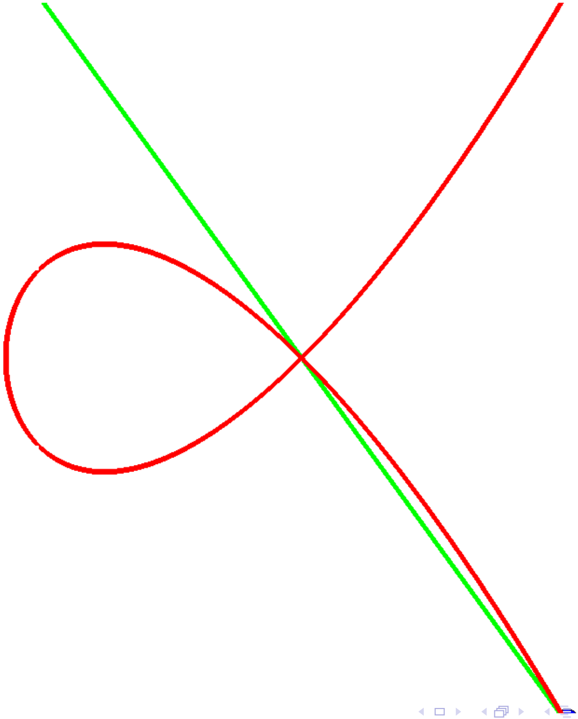


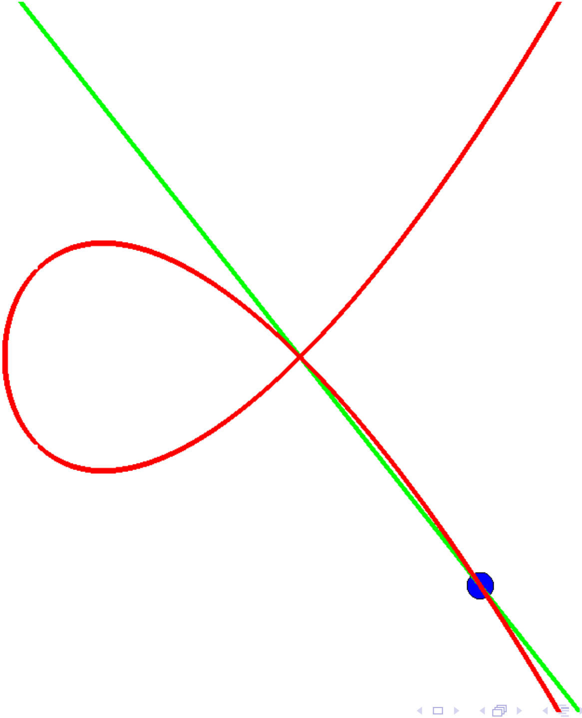


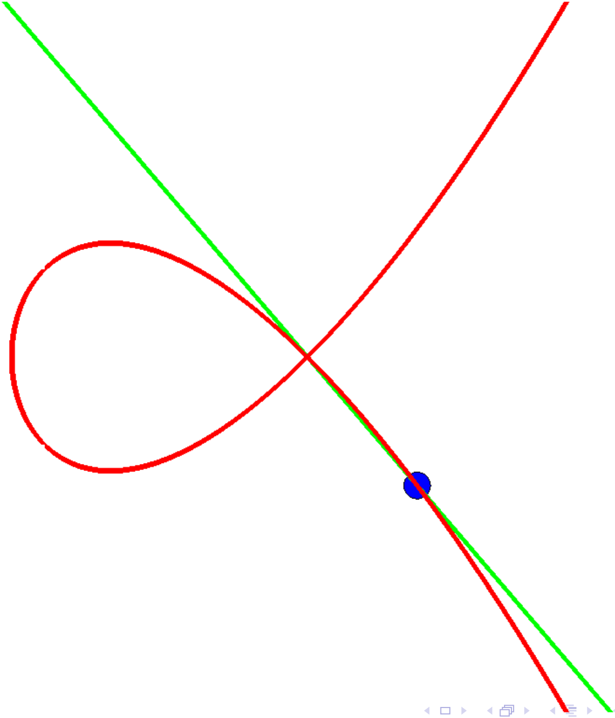


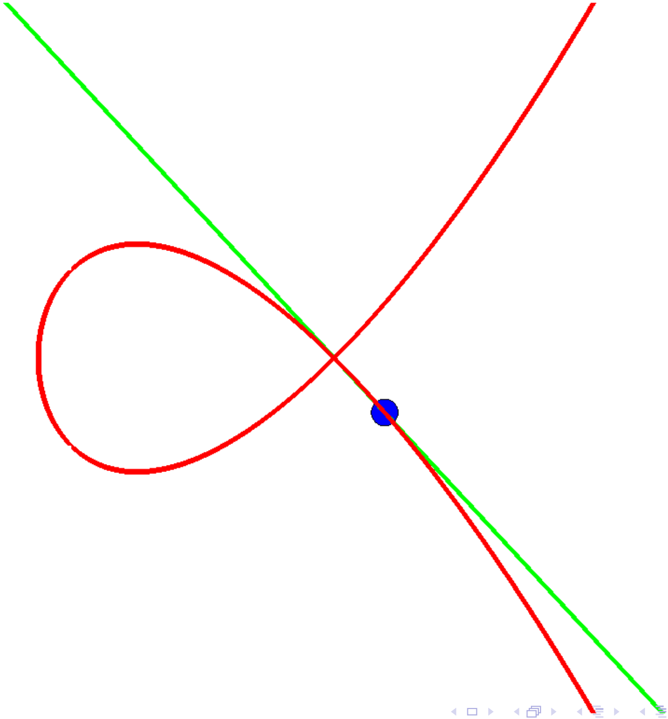


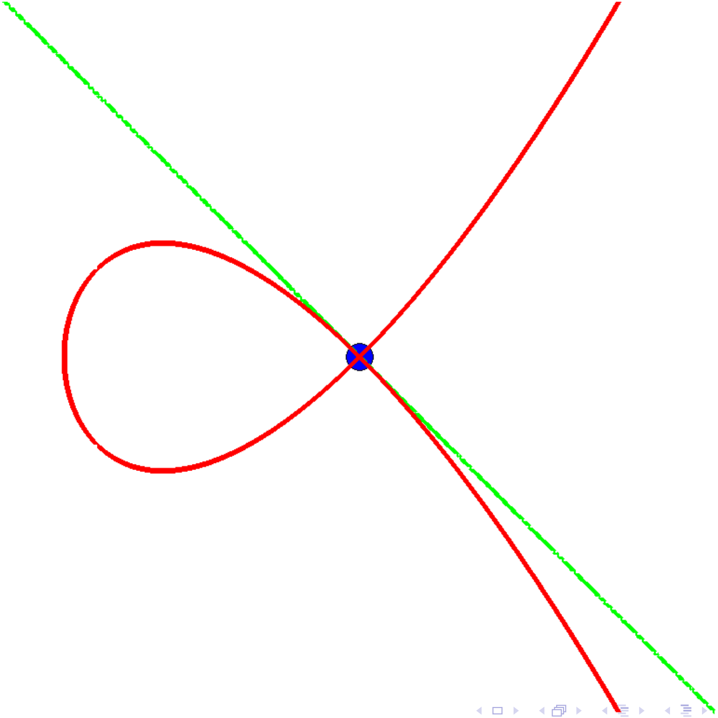


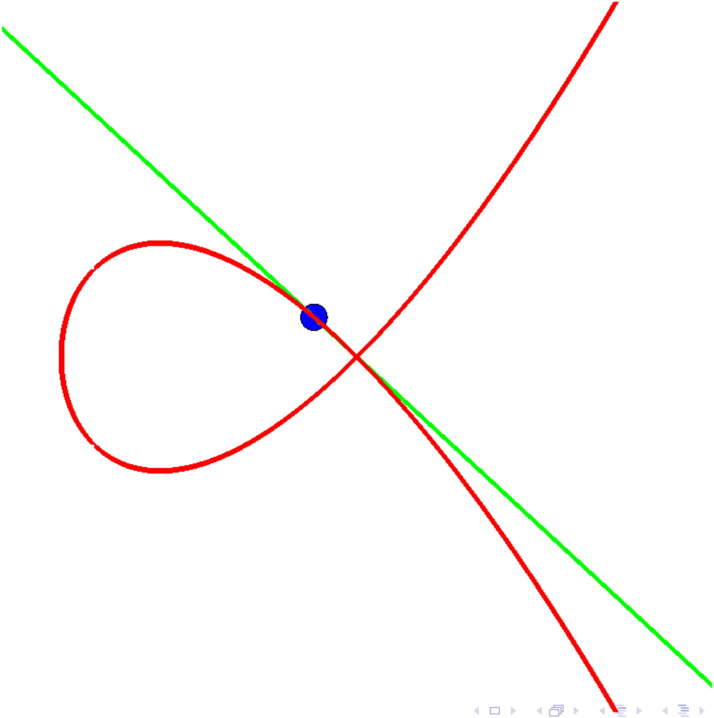


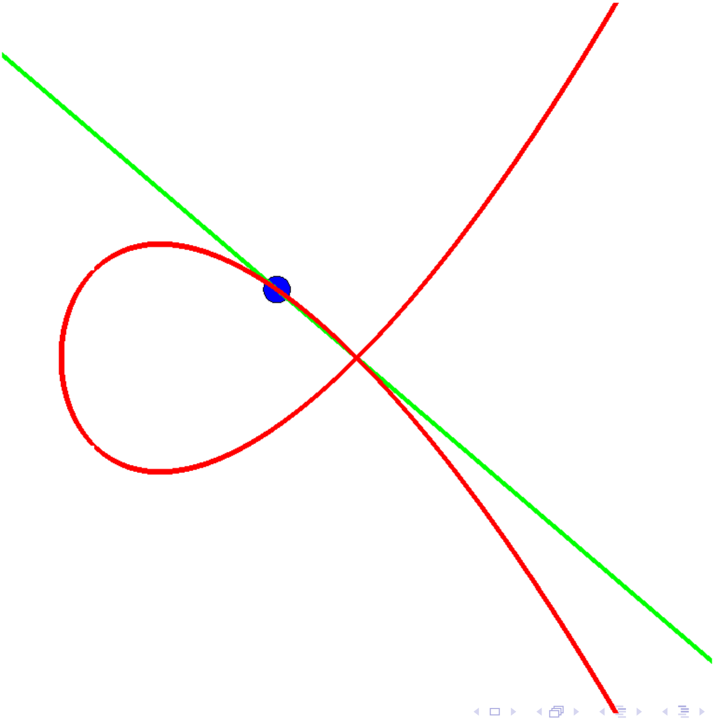


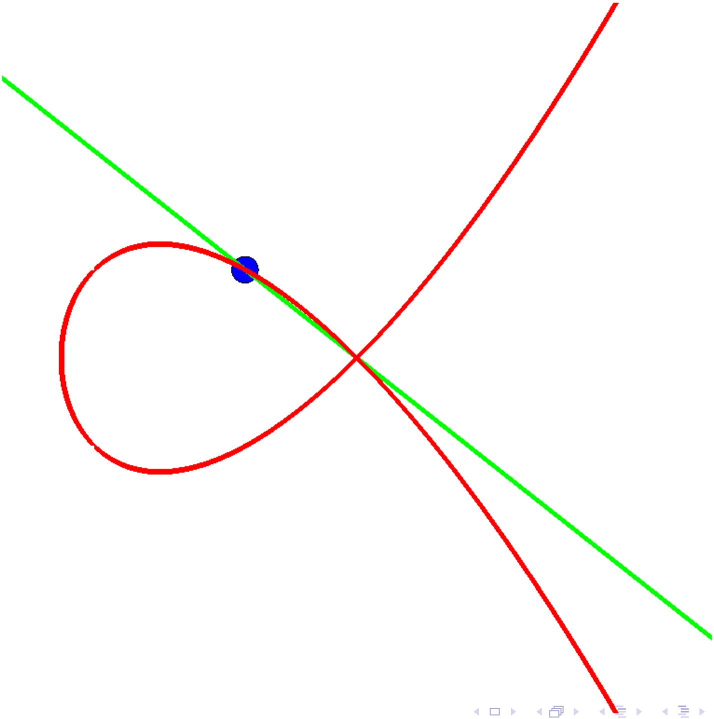


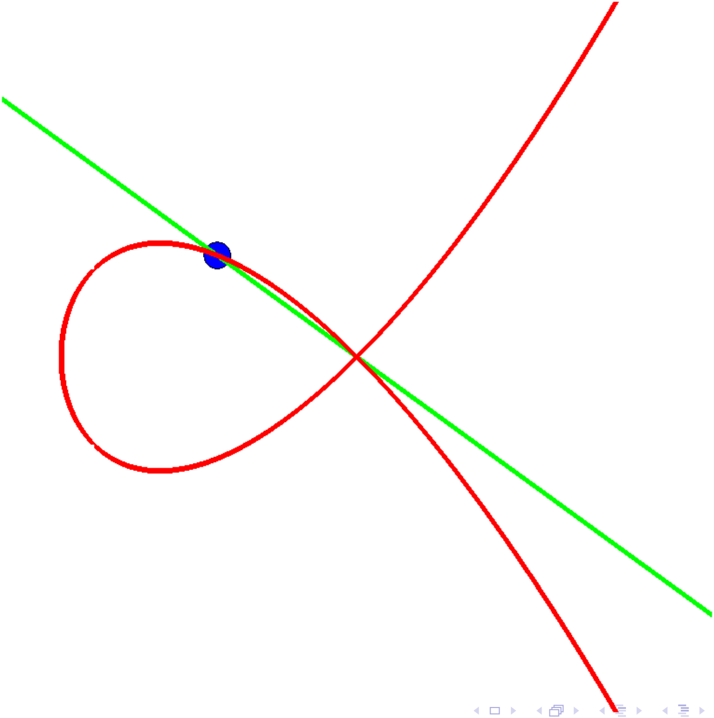


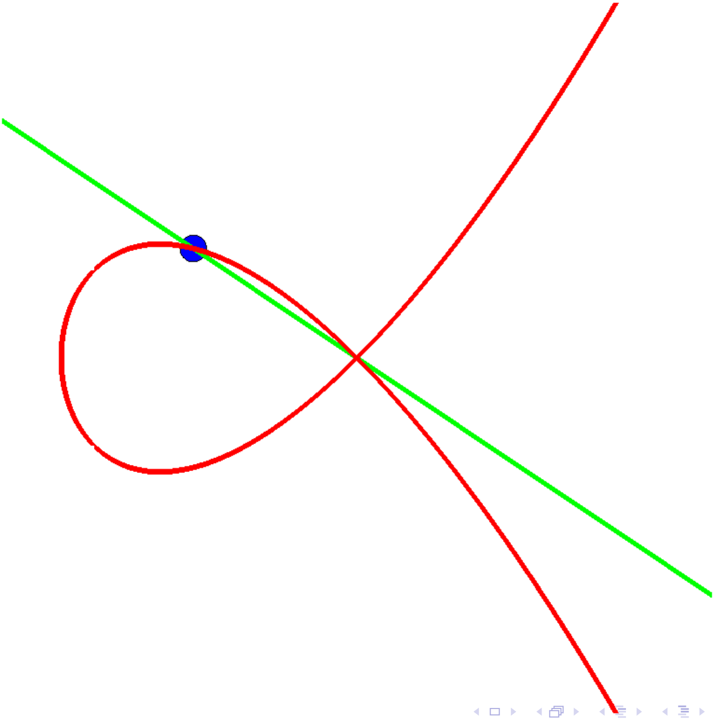


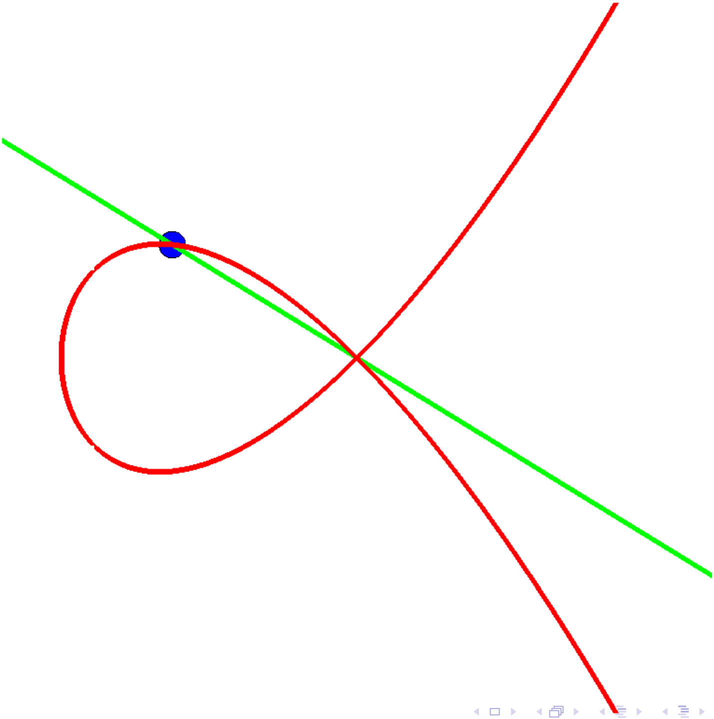


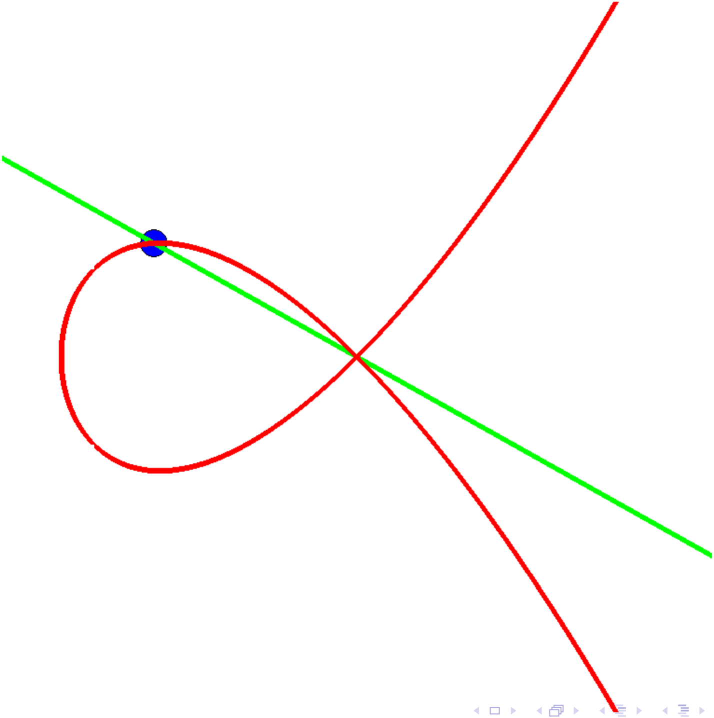


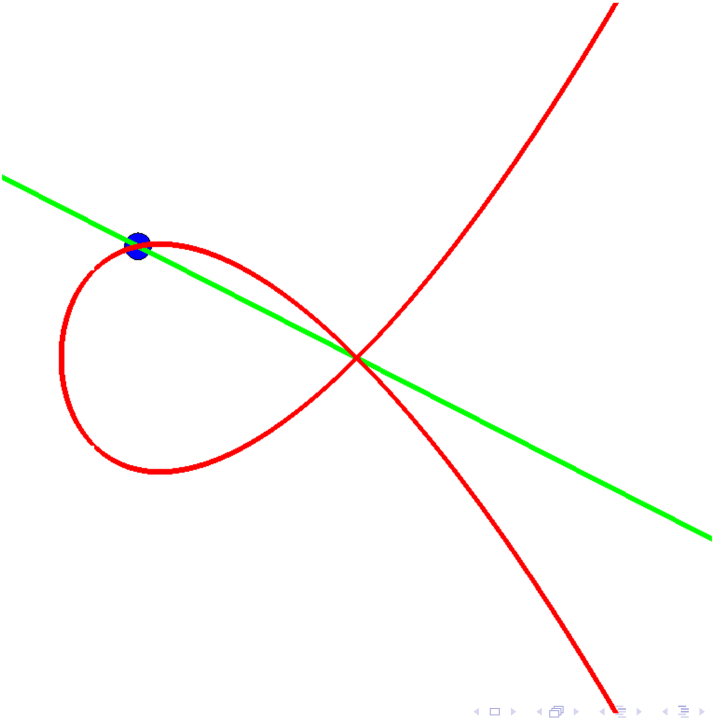


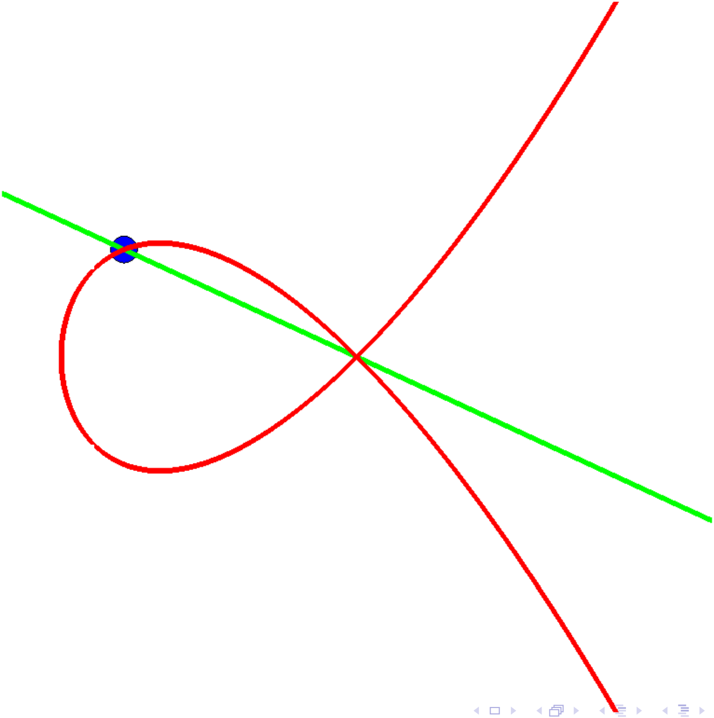


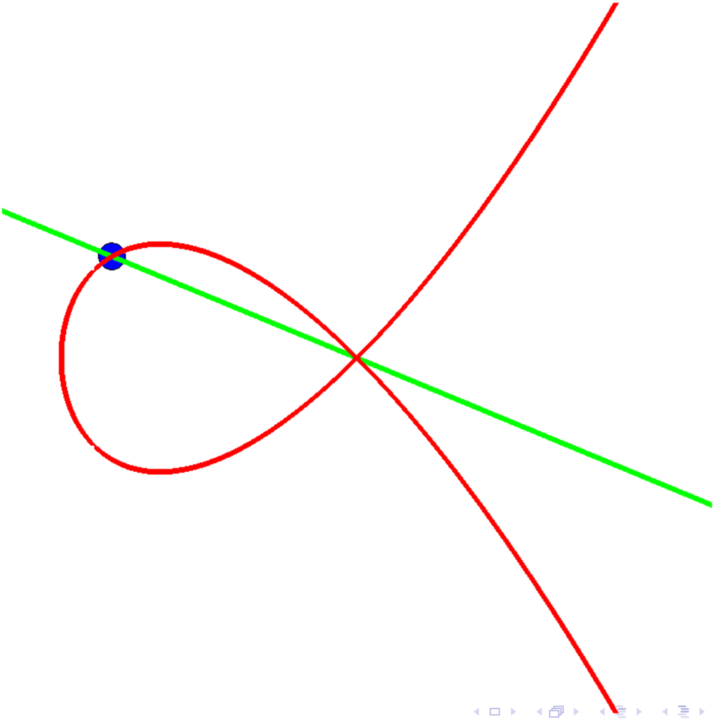


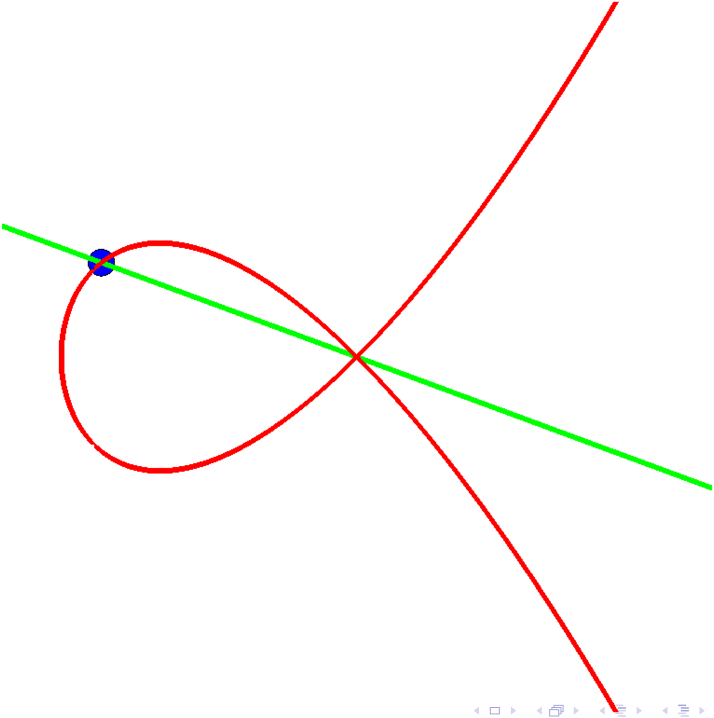


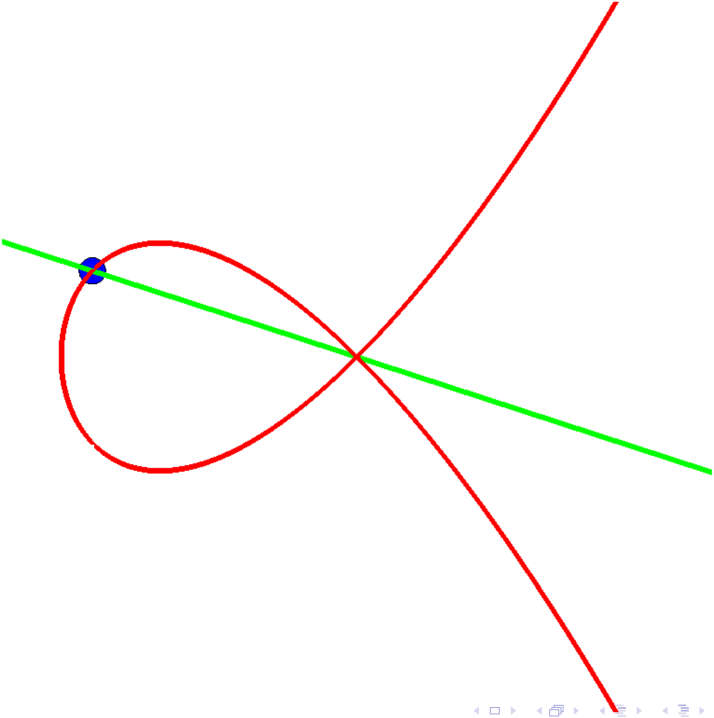


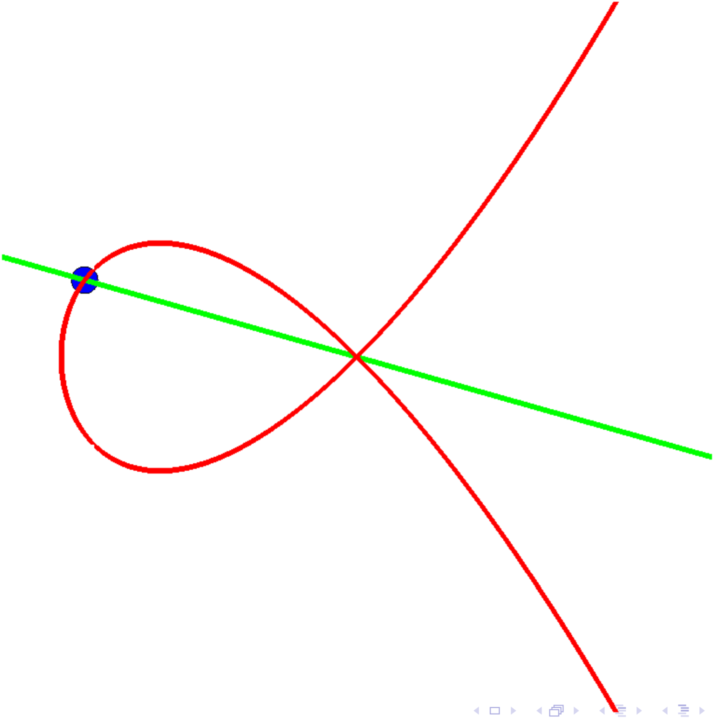


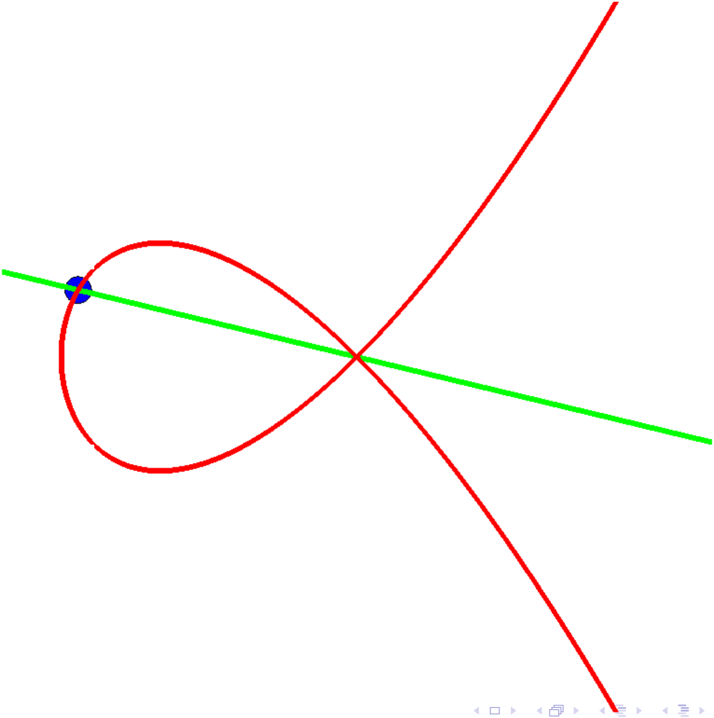


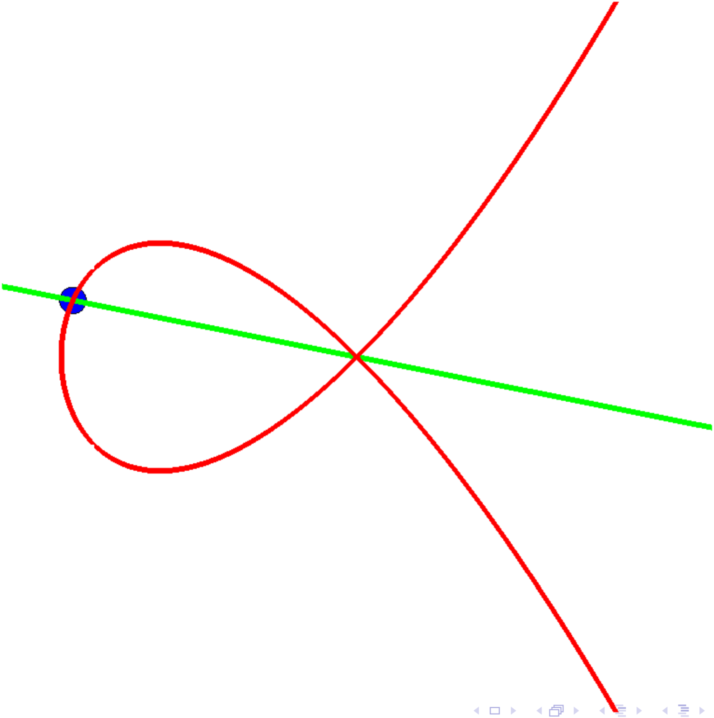


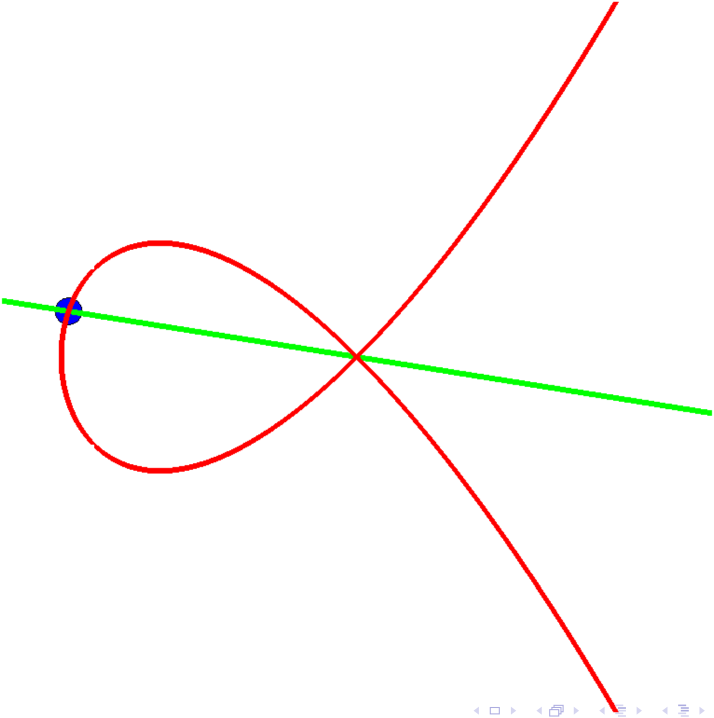


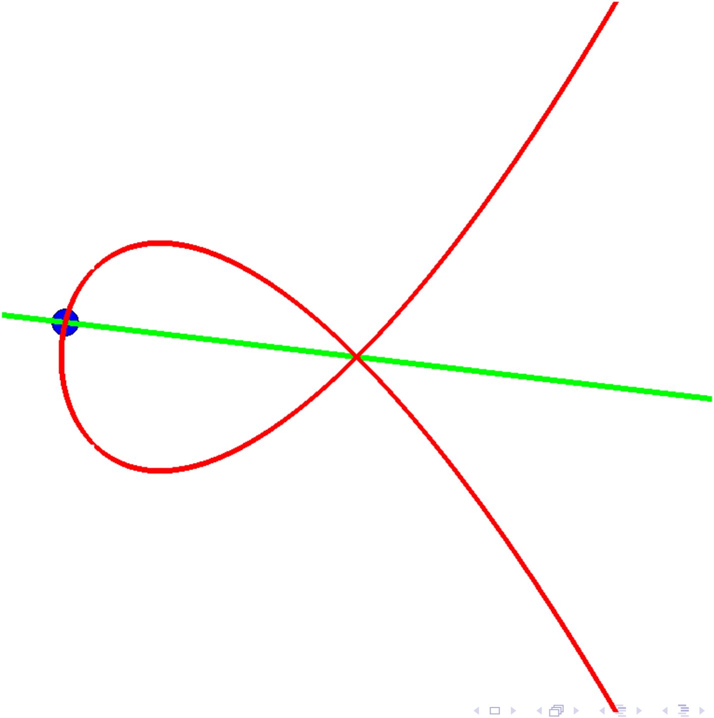


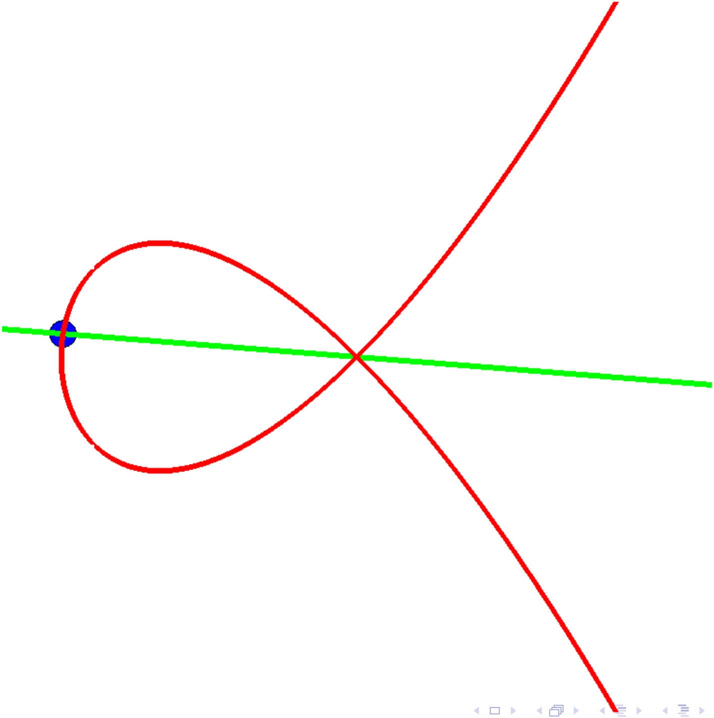


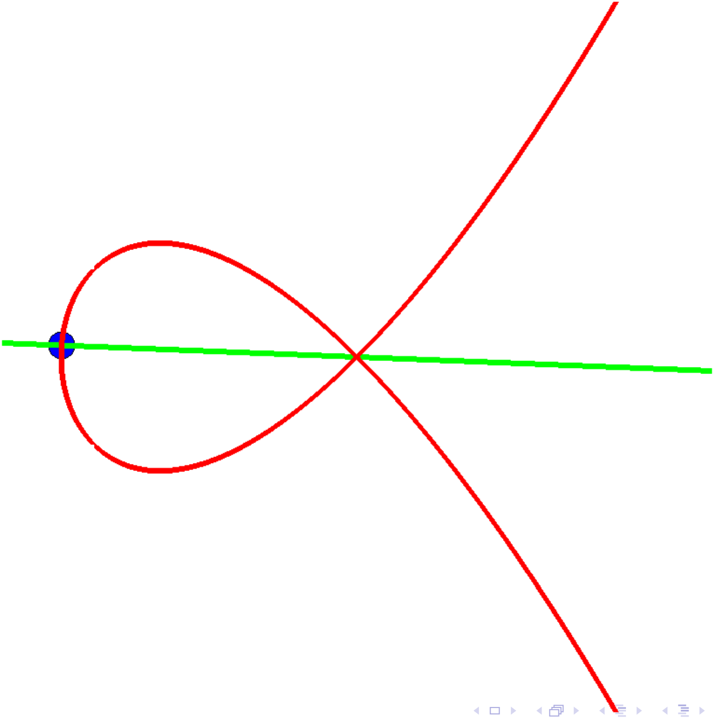


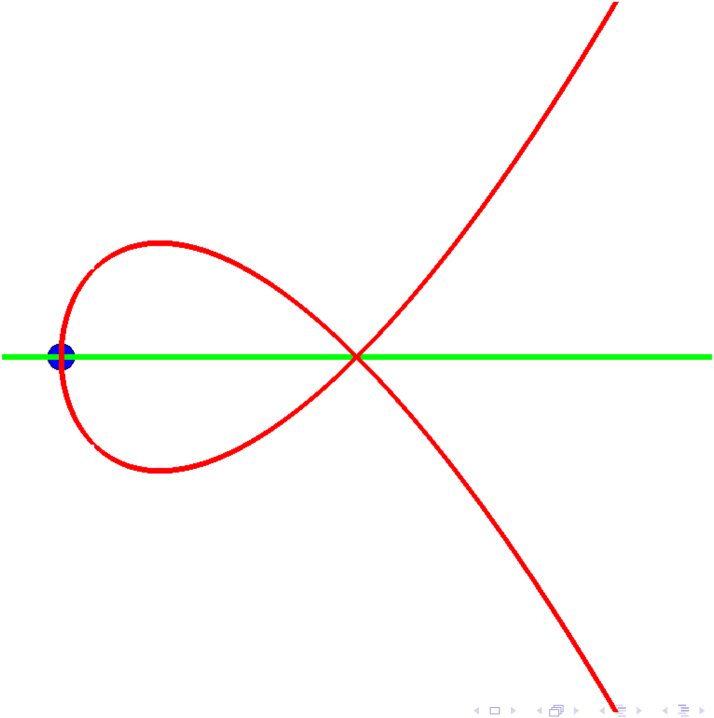


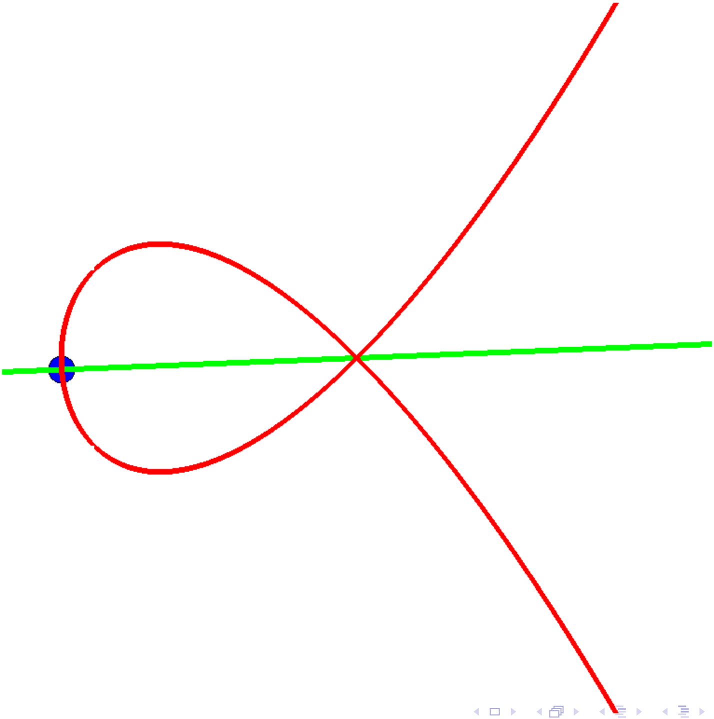


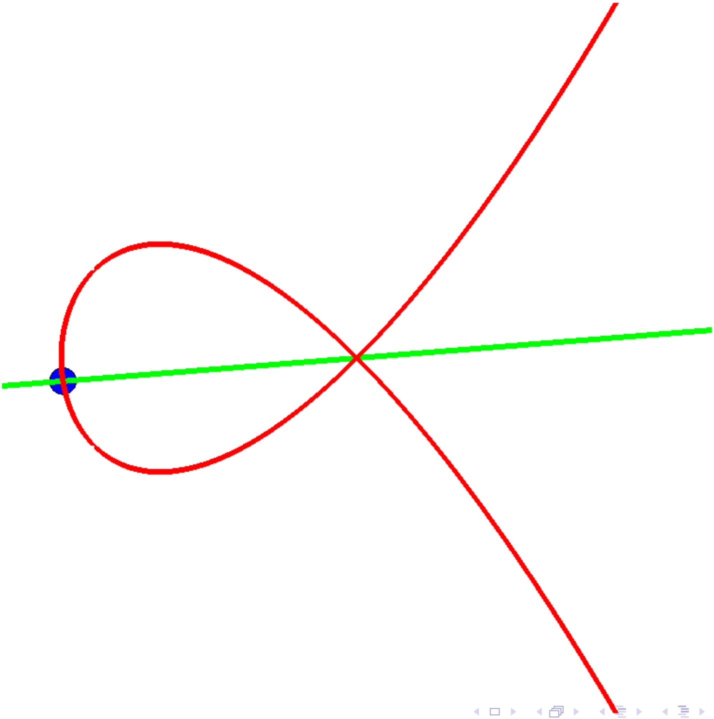


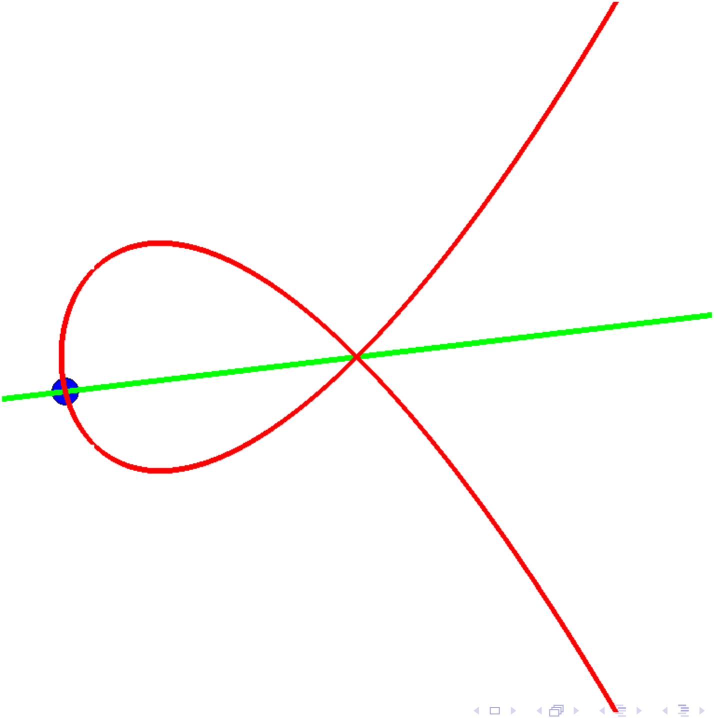


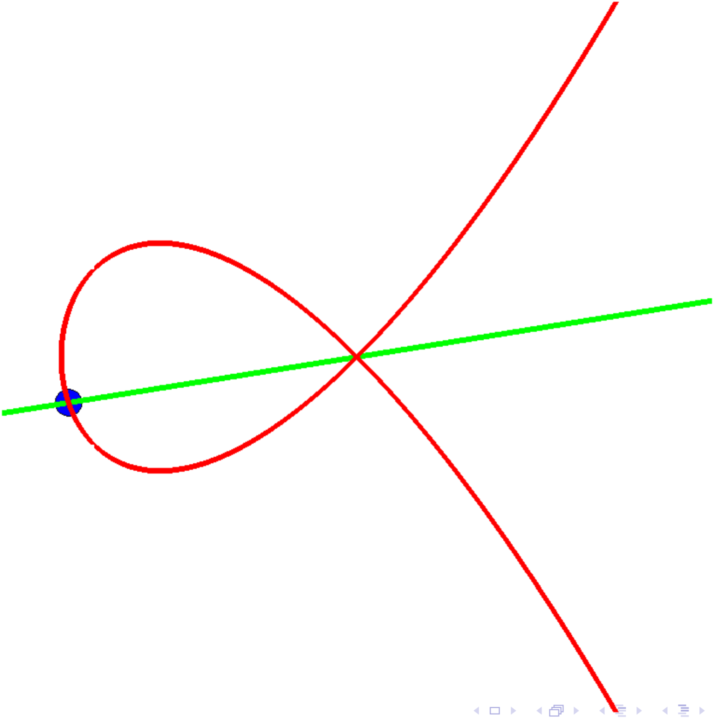


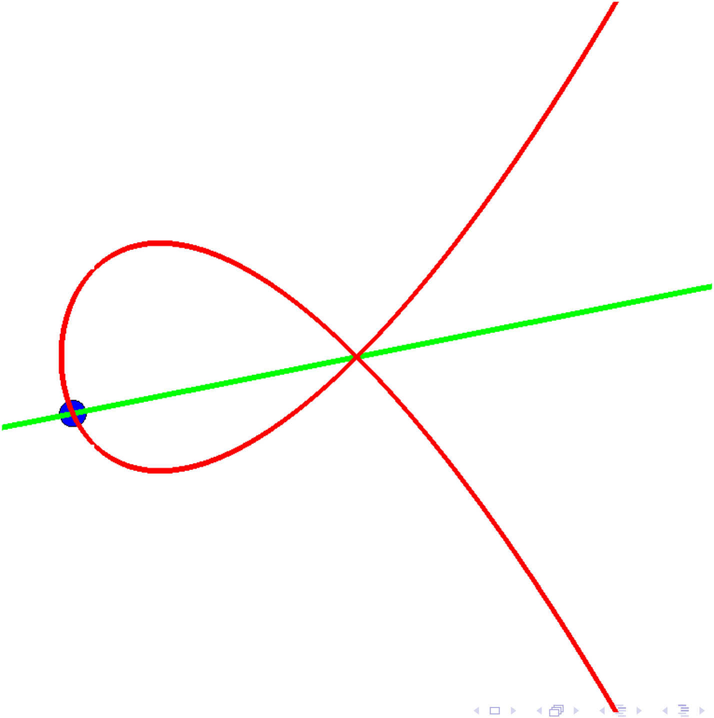


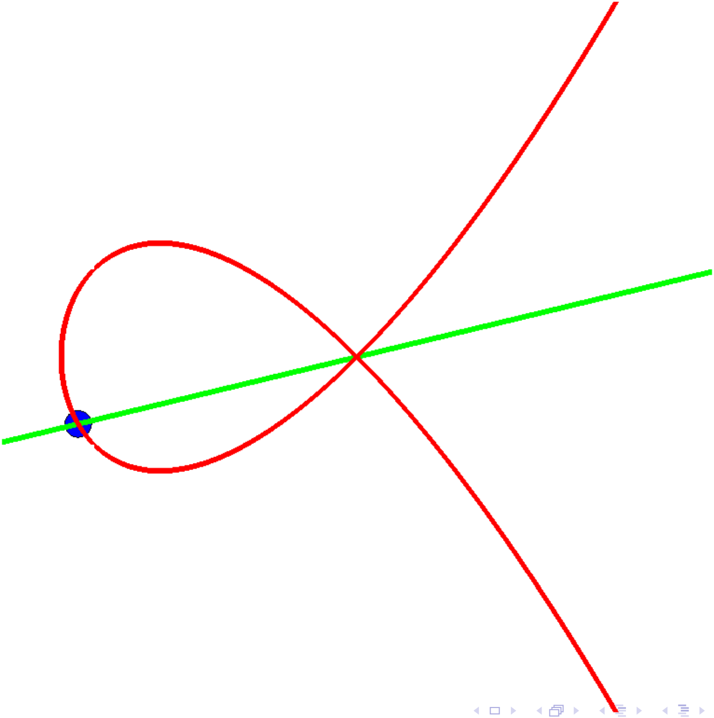


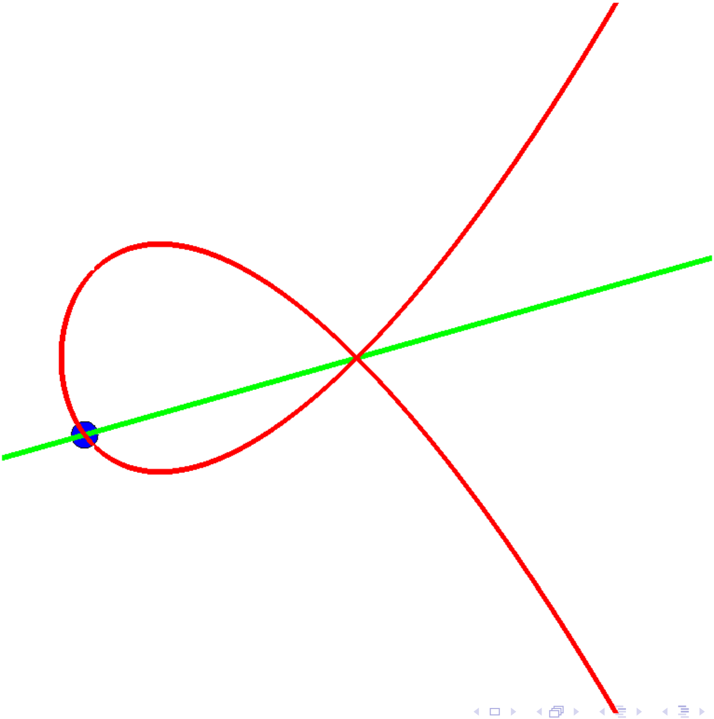


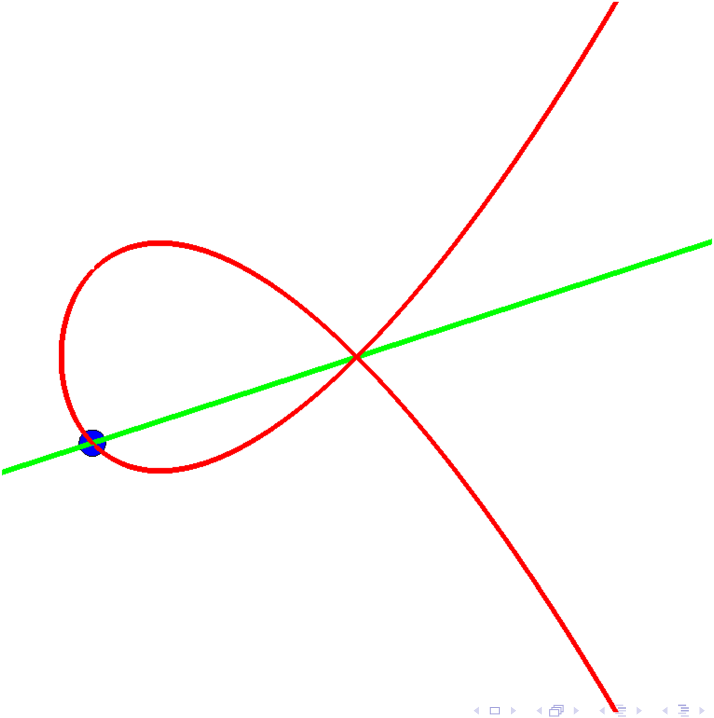


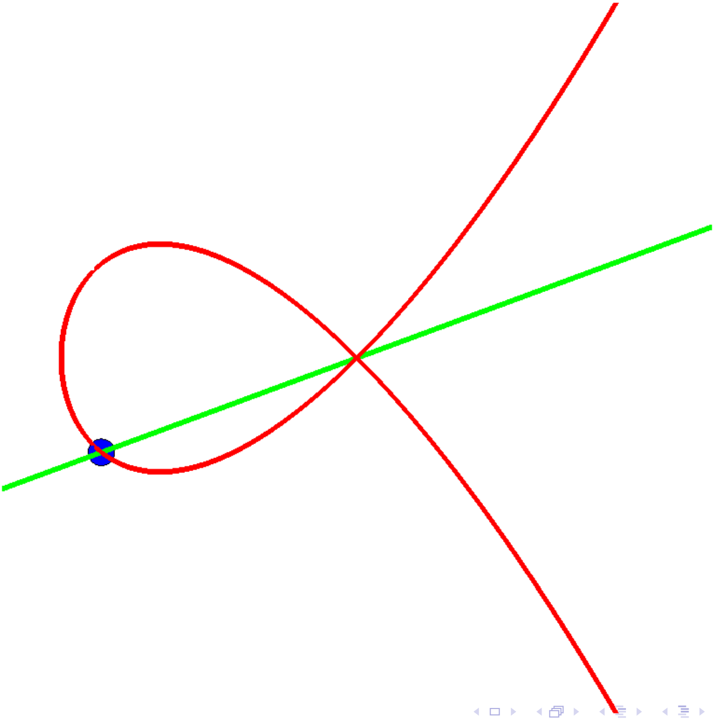


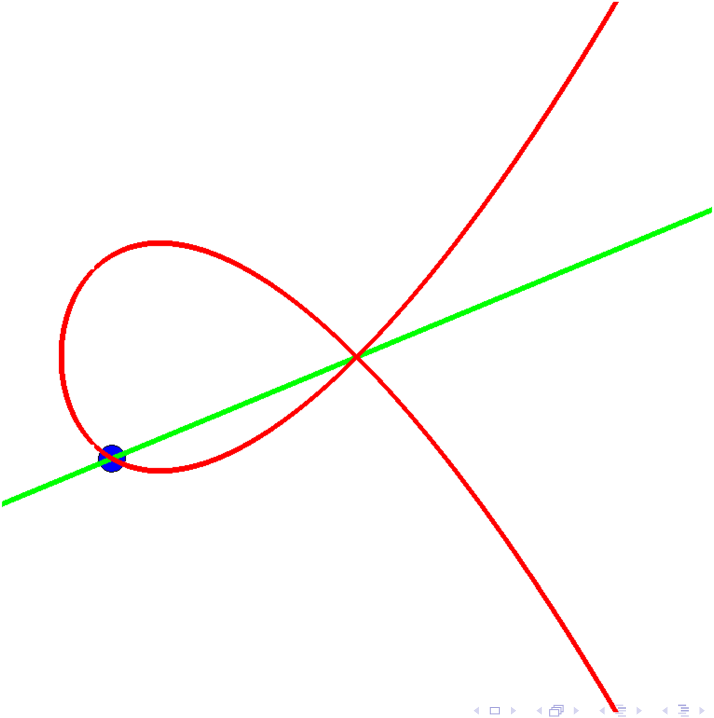


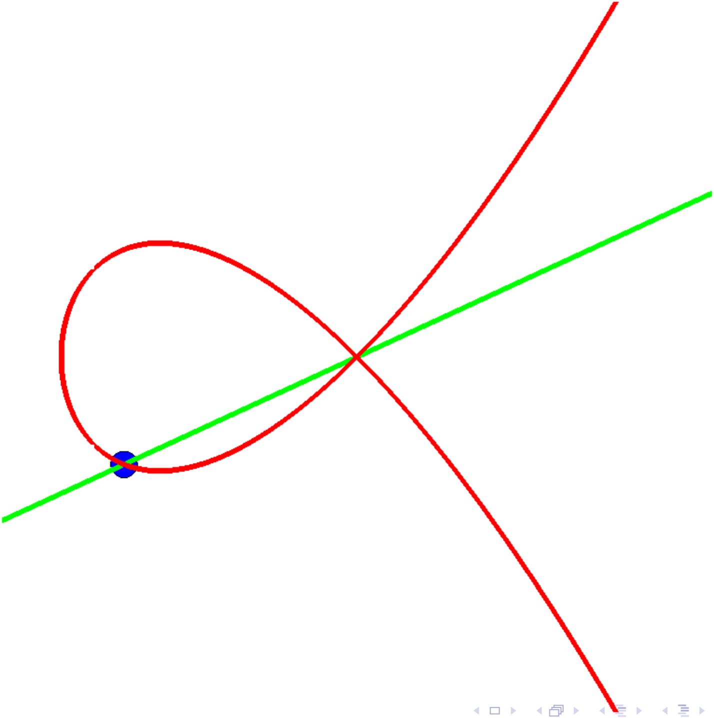


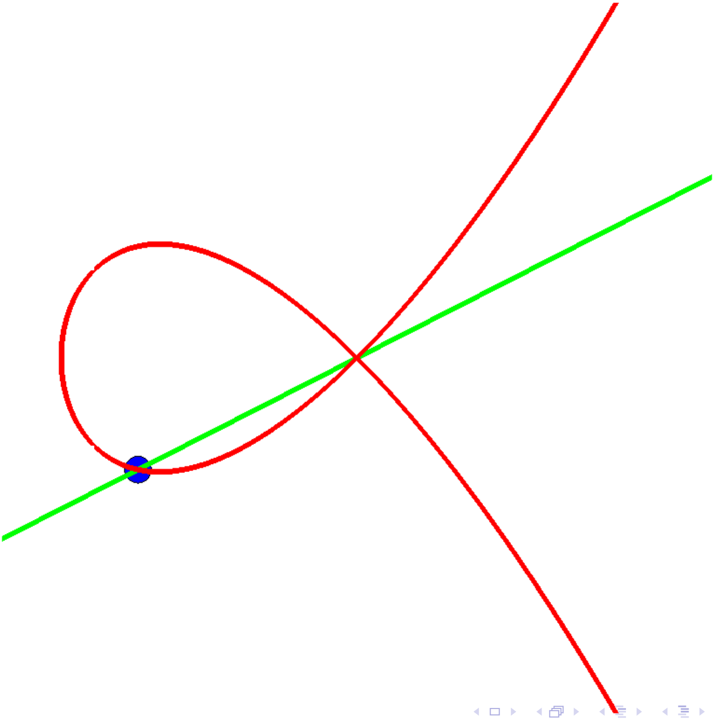


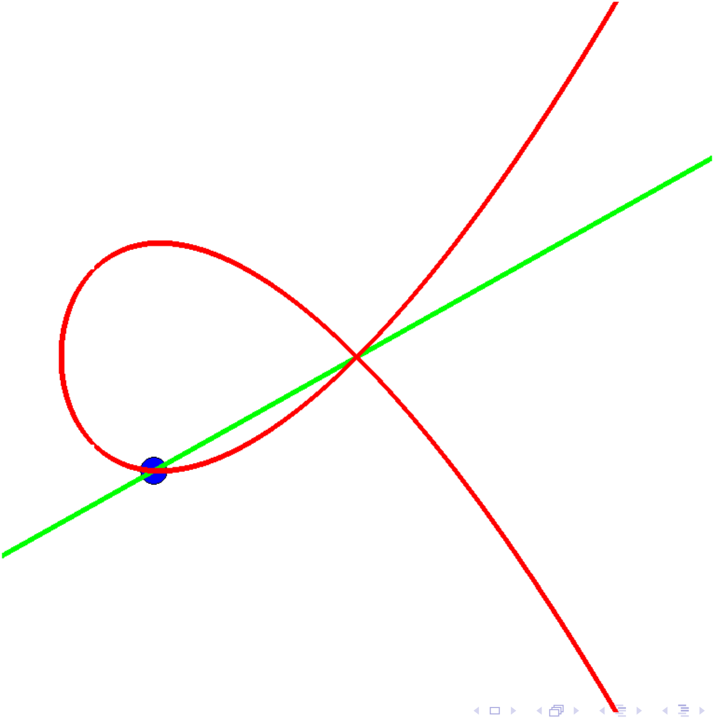


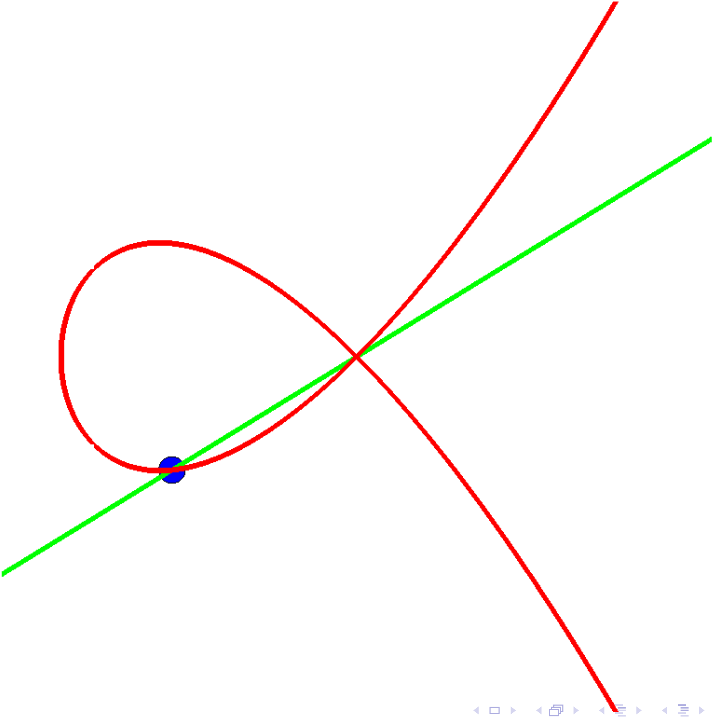


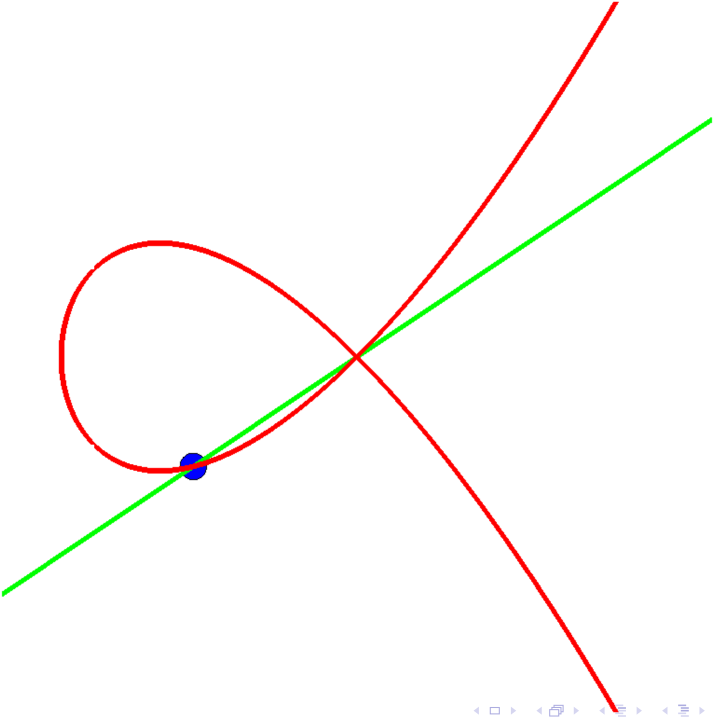


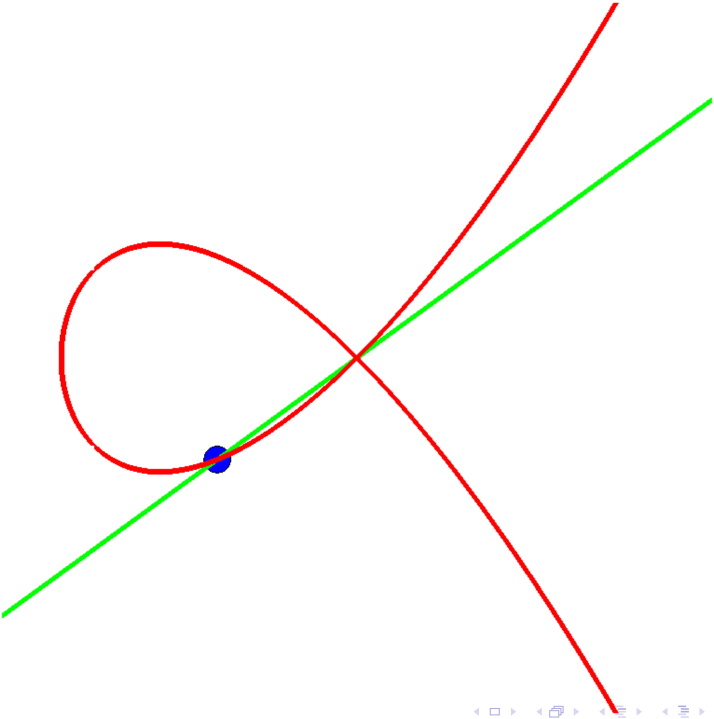


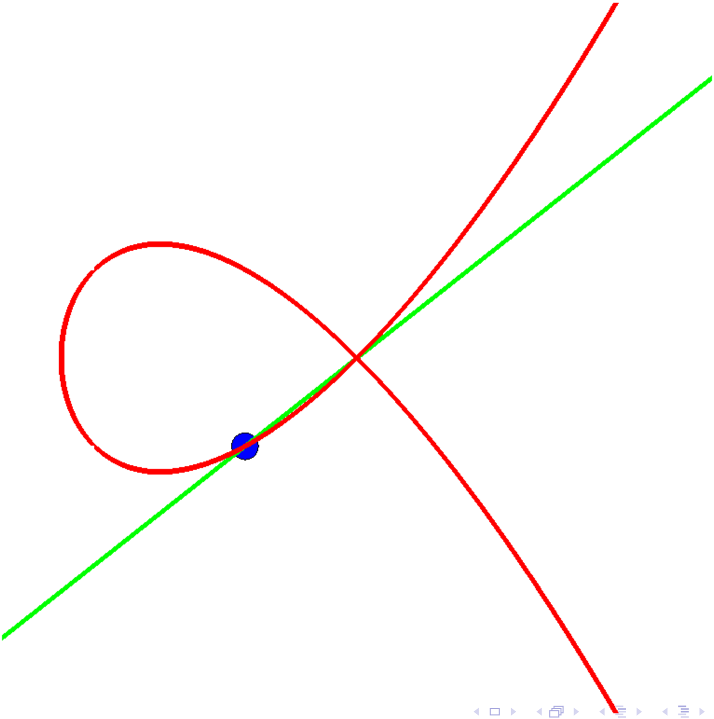


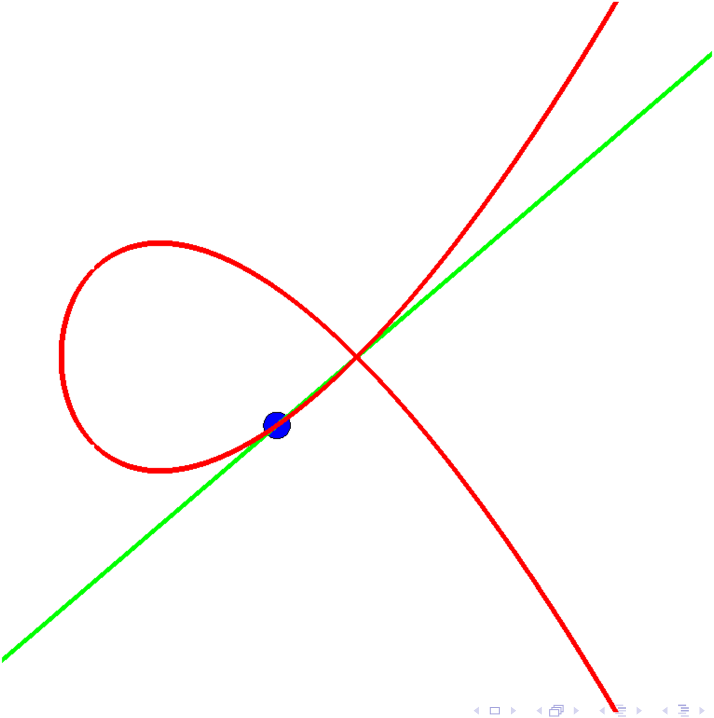


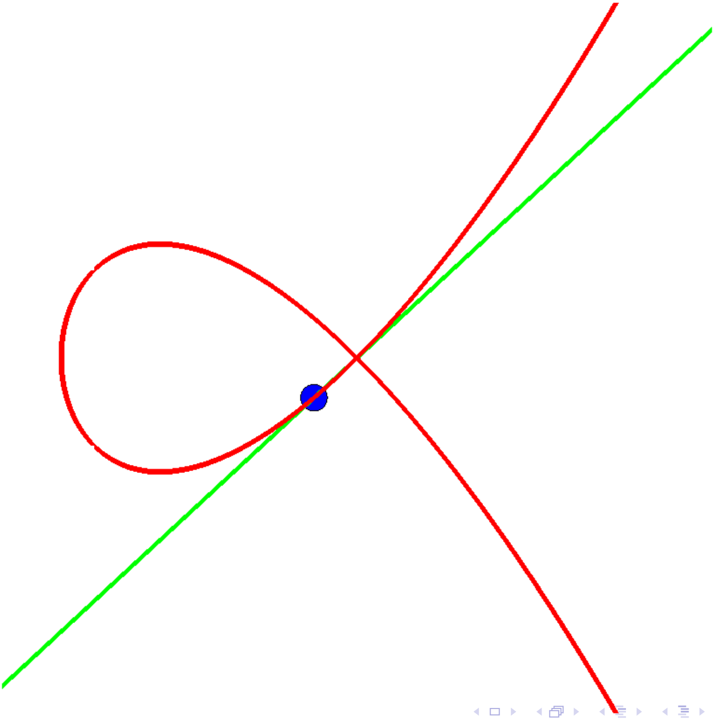


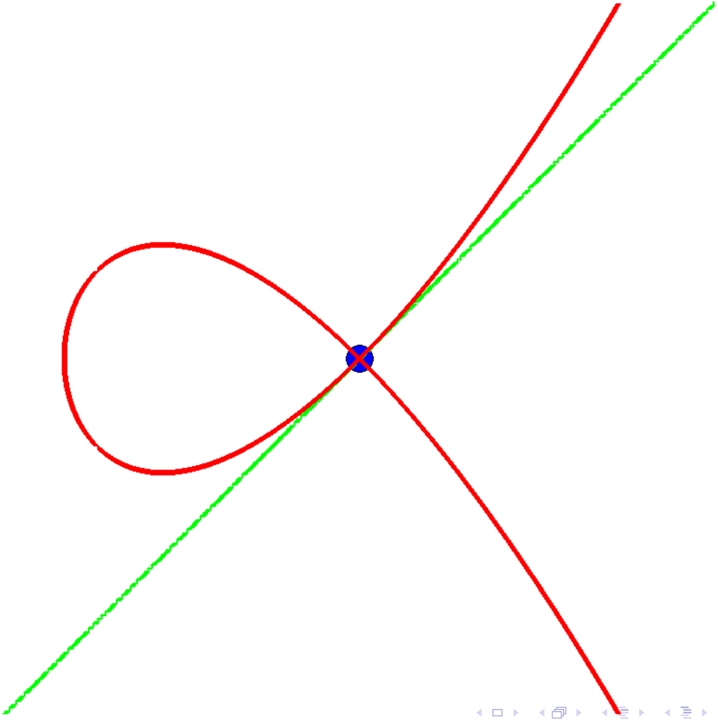


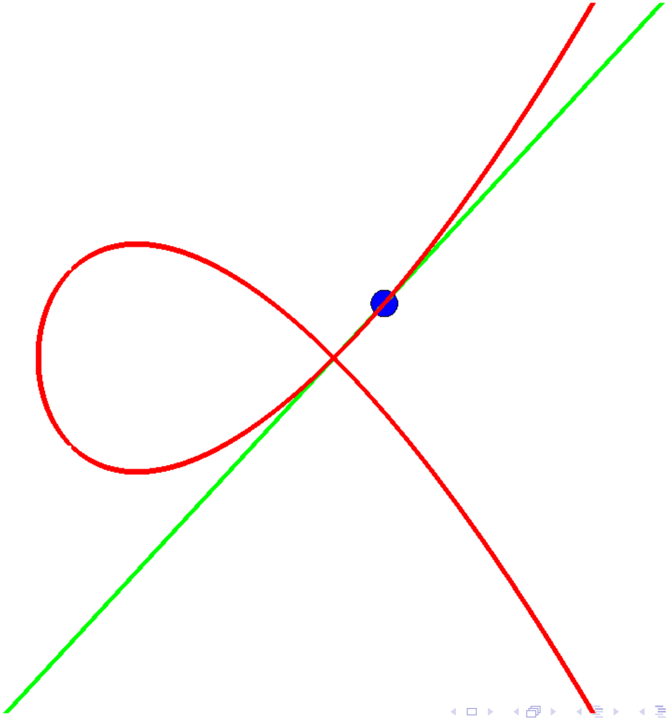


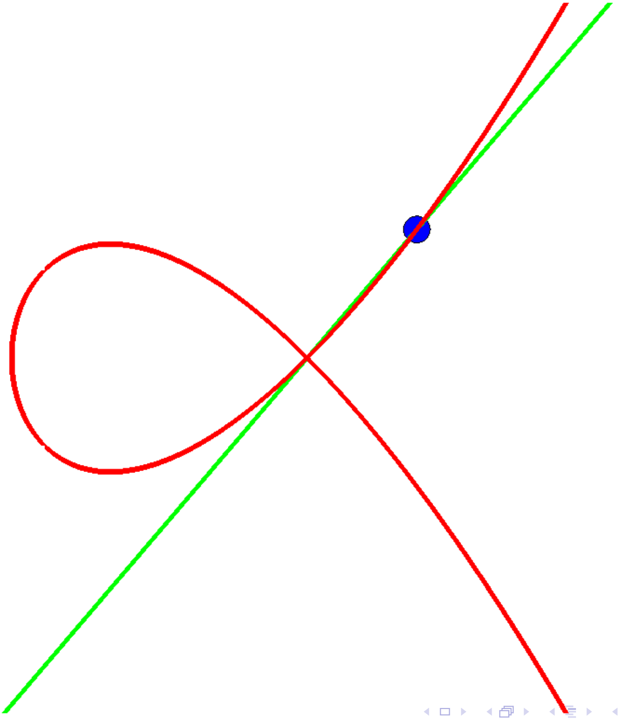


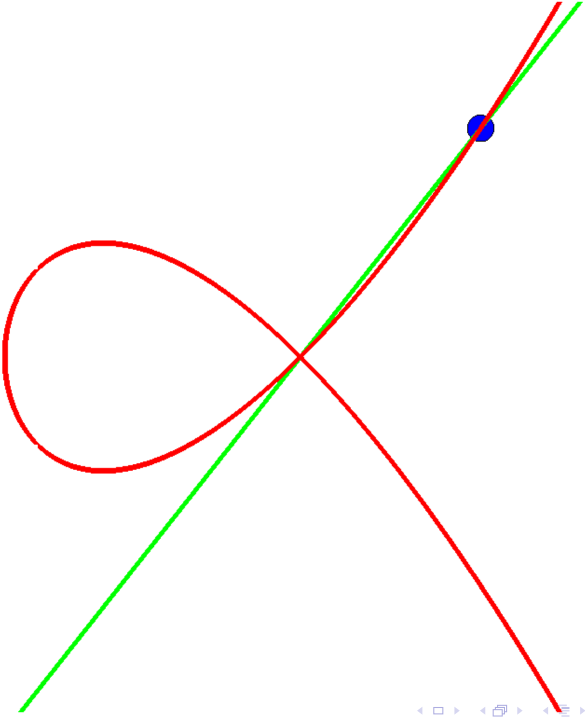


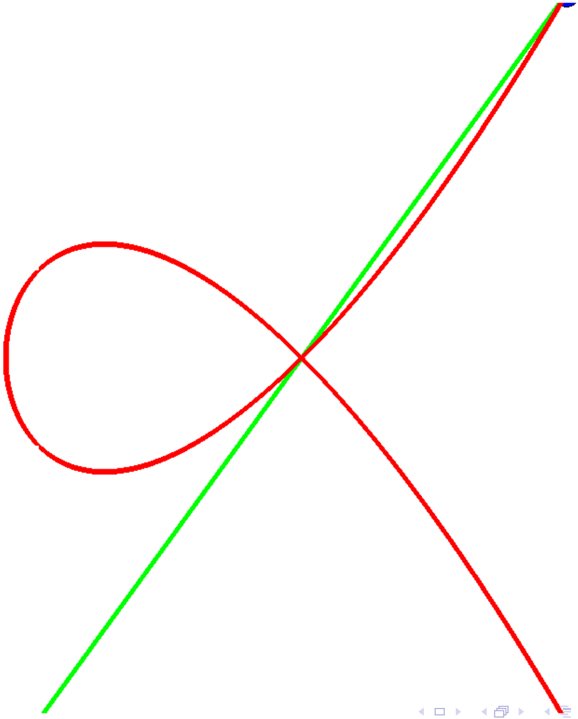


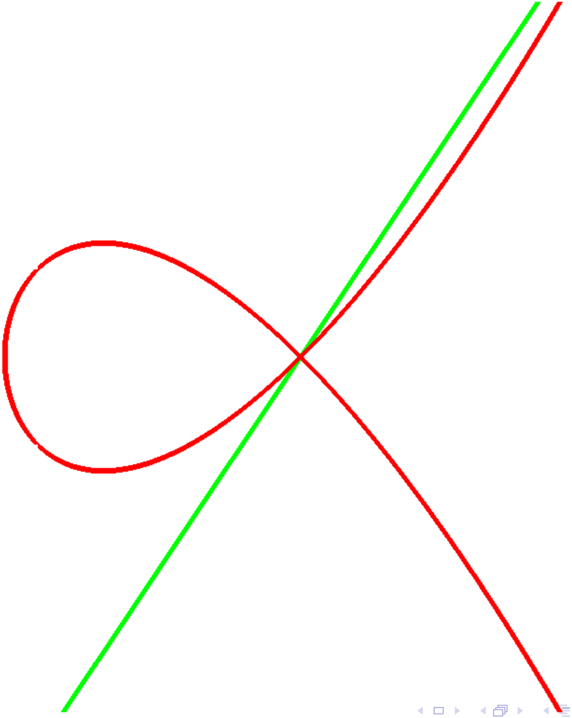


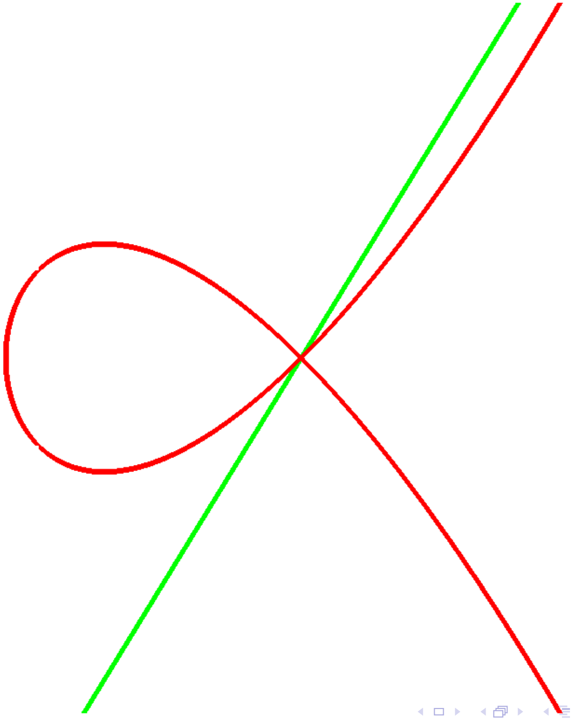


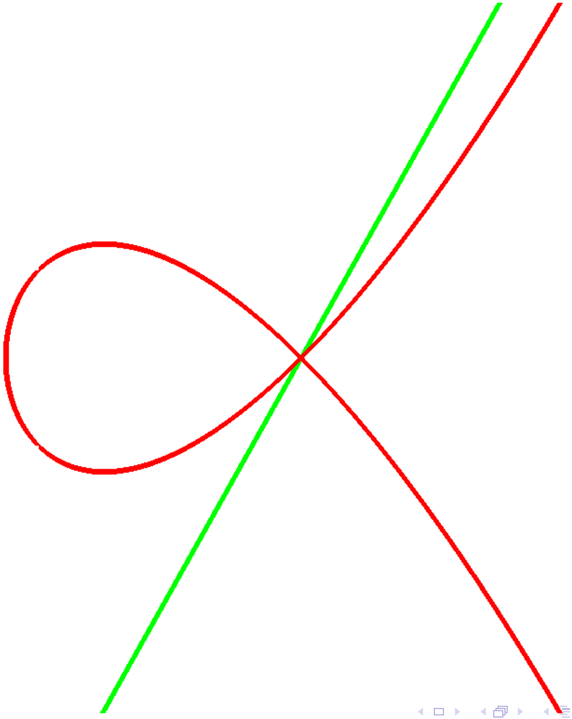


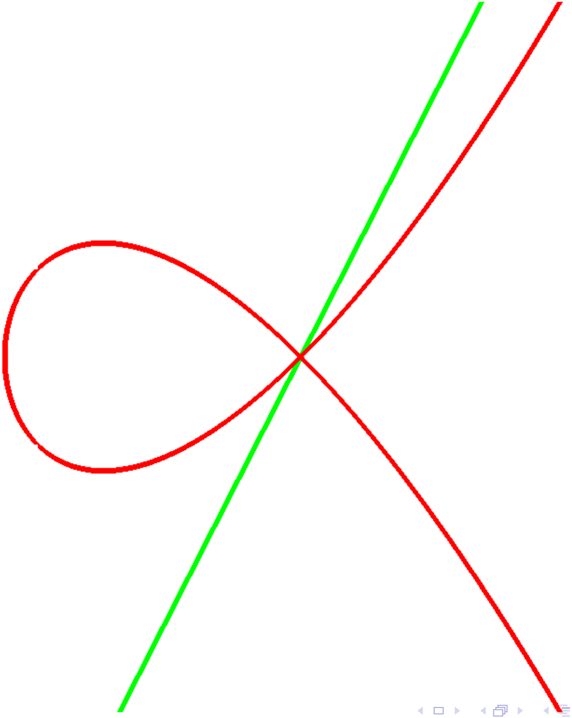


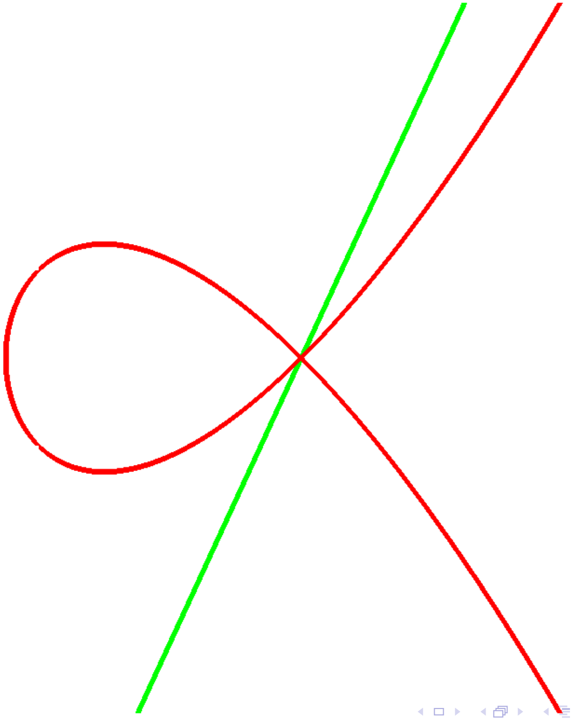


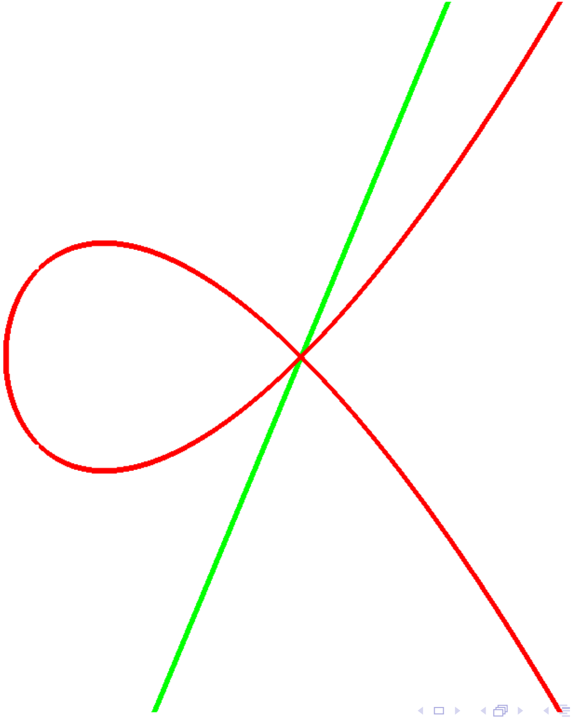


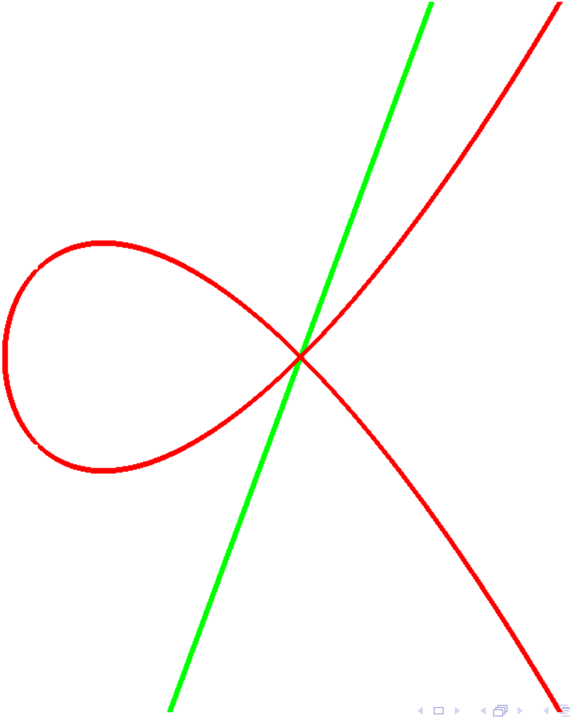


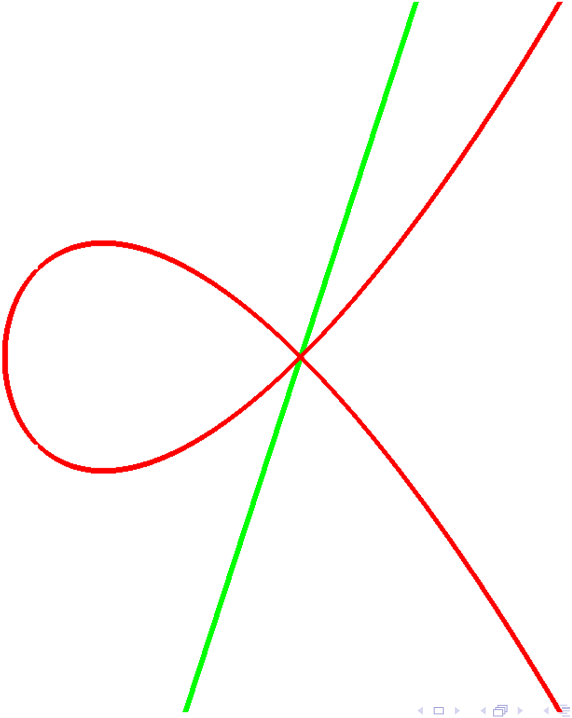


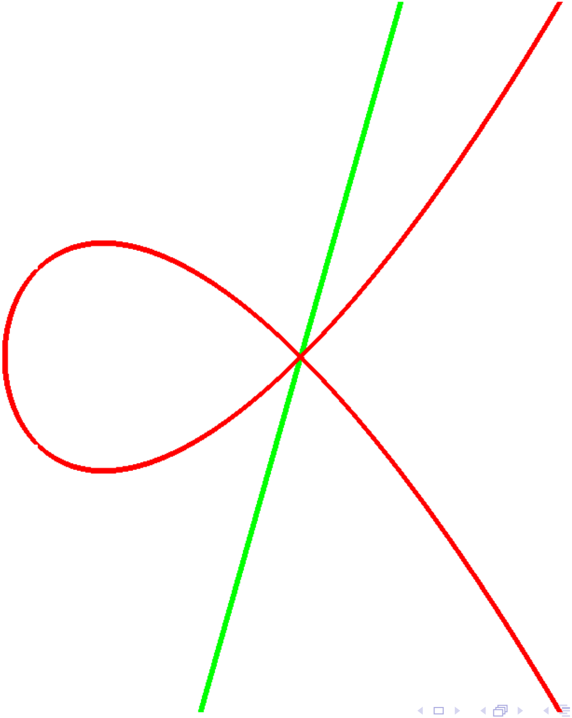


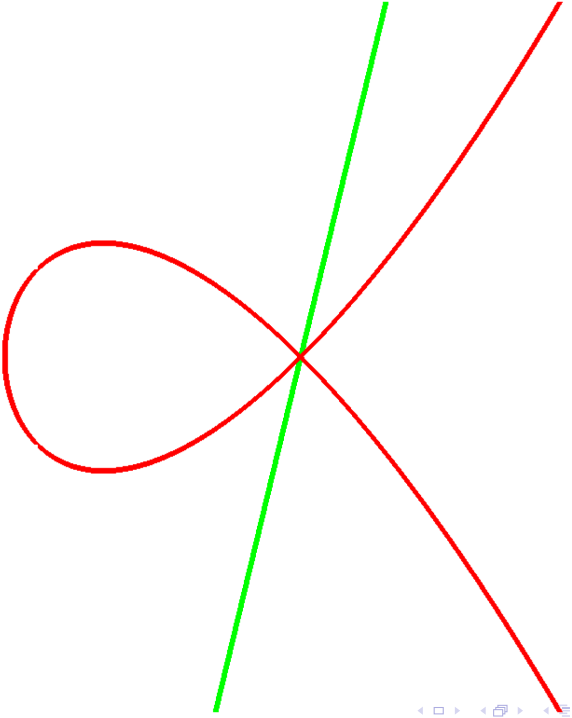


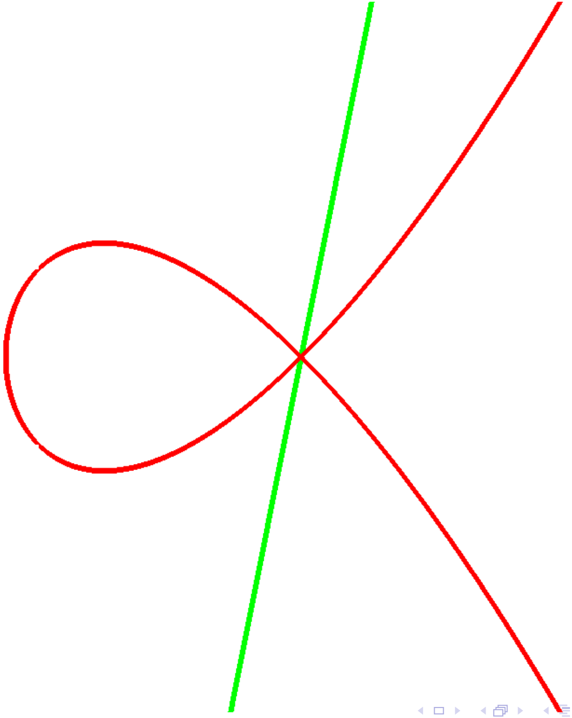


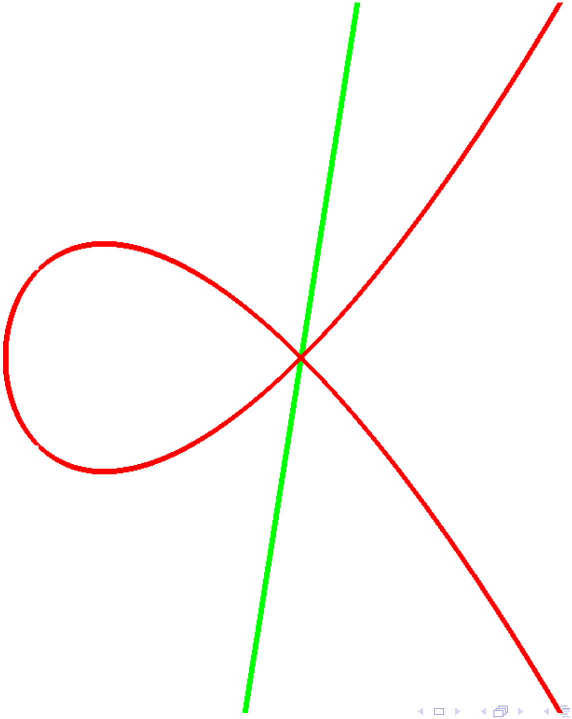


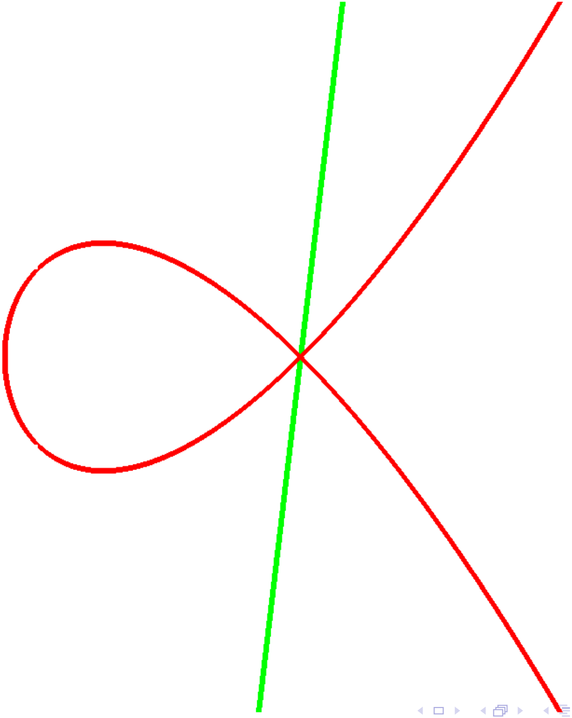


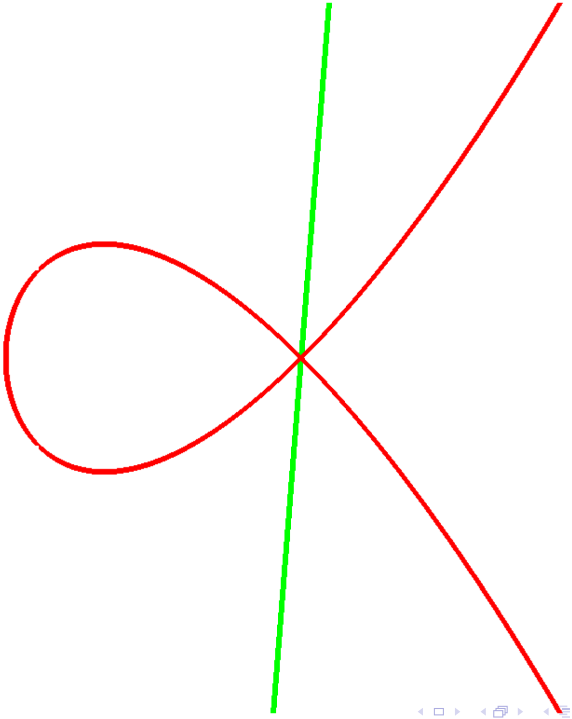


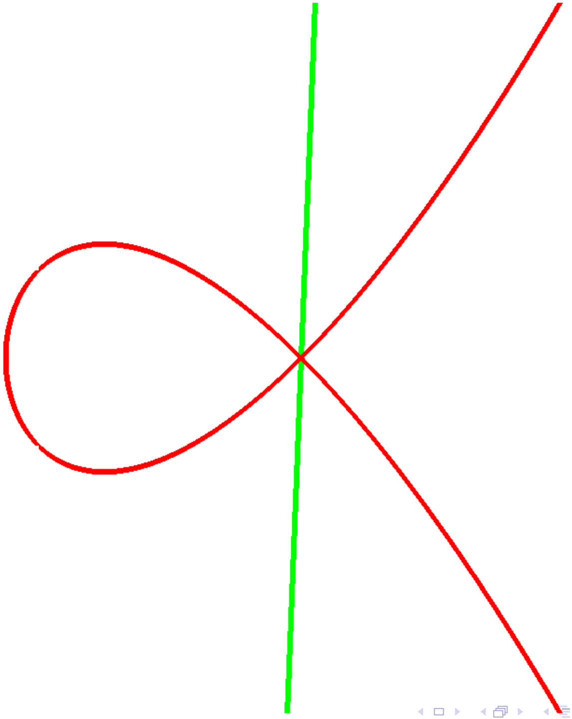


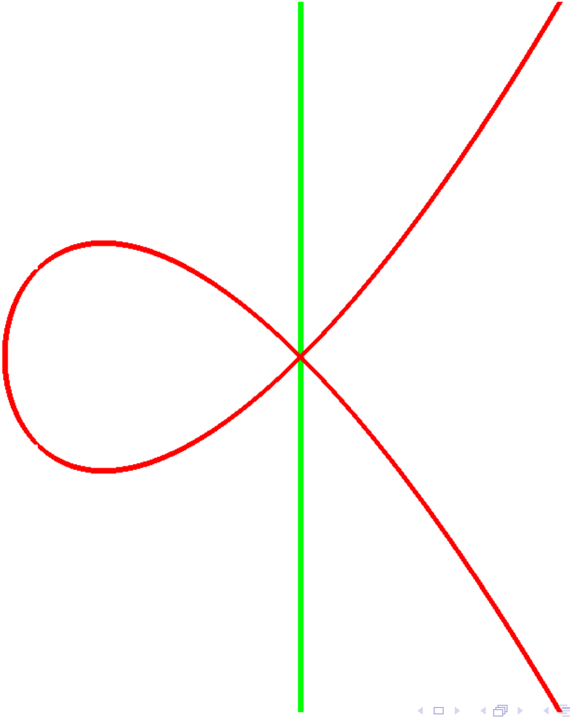












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Theorem (Noether)

$\overline{A}$ is a finitely generated A-module.

Example

For $C = V(I)$ where $I = \langle x^3 + x^2 - y^2 \rangle \subset K[x, y]$

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So if p is a prime divisor of s, then also of r, which implies that $\frac{r}{s} \in A$. $\square$

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If $J \subset A$ is an ideal and $0 \neq g \in J$, then

Key Lemma

Definition

For ideals $I_j \subset A$ the **ideal quotient** is $(I_1 :_A I_2) = \{b \in A \mid bI_2 \subset I_1\}$.

Algorithm (Ideal quotients by Elimination)

$I : \langle g_1, g_2 \rangle = (I : g_1) \cap (I : g_2)$
$I \cap \langle g \rangle = \langle g \cdot f_1, ..., g \cdot f_s \rangle \Longrightarrow I : \langle g \rangle = \langle f_1, ..., f_s \rangle$
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Lemma

If $J \subset A$ is an ideal and $0 \neq g \in J$, then

$$\begin{array}{ccccccc} A & \hookrightarrow & \mathrm{Hom}_A(J, J) & \cong & \frac{1}{g}(gJ :_A J) & \subset & \overline{A} \\ a & \mapsto & a\cdot & & & & \\ & & \varphi & \mapsto & \frac{\varphi(g)}{g} & & \end{array}$$

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$$\sum_{i=0}^{s} \frac{a_i}{g^i} \left(\frac{b}{g} \right)^{s-i} = 0 \quad \text{with} \quad \frac{a_i}{g^i} \in A.$$

□

Non-normal and singular locus

For simplicity, assume $V(I)$ is curve. Then the **non-normal locus** $N(A)$ is equal to the **singular locus** $\operatorname{Sing}(A)$.

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$$\text{Sing}(A) = V(\text{Jac}(I) + I).$$

Example

For $I = \langle x^4 + x^5 - y^2 \rangle$ we have $\text{Jac}(I) + I = \langle x^3, y \rangle$.

Grauert-Remmert Criterion

Definition

The **radical** of an ideal $I \subset A$ is

$$\sqrt{I} = \{f \in A \mid \exists a \in \mathbb{N} \text{ with } f^a \in I\}$$

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Let $0 \neq J \subset A = K[x_1, ..., x_n]/I$ *be an ideal with* $J = \sqrt{J}$ *and*

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Then A *is normal iff the inclusion*

$$\begin{array}{ccc} A & \hookrightarrow & \mathrm{Hom}_A(J, J) \\ a & \mapsto & (b \mapsto ab) \end{array}$$

is an isomorphism.

Normalization Algorithm

If A is not normal, then for $J = \sqrt{\mathrm{Jac}(I) + I}$ by Grauert-Remmert

$$A \subsetneqq \mathrm{Hom}_A(J, J) \cong \frac{1}{g}(gJ :_A J) \subset \overline{A} \subset Q(A).$$

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Lemma

$N(A_i) \subset V(J_i)$

Normalization Algorithm

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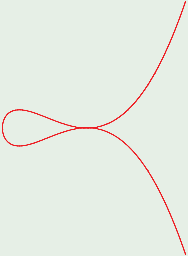

Normalization Algorithm

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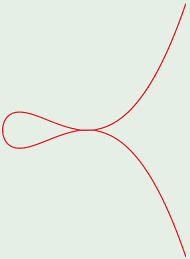

the first step yields

$$A_1 = \frac{1}{x}\,(xJ : x) = \frac{1}{x}\,\langle x, y\rangle =_A \left\langle 1, \frac{y}{x} \right\rangle$$

Normalization Algorithm

To obtain $\overline{A}$, one more iteration is needed, leading to $A_2 =_A \left\langle 1, \frac{y}{x^2} \right\rangle$:

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  [1]:
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Desingularization of Curves by Normalization

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$$t = \frac{y}{x^2}$$

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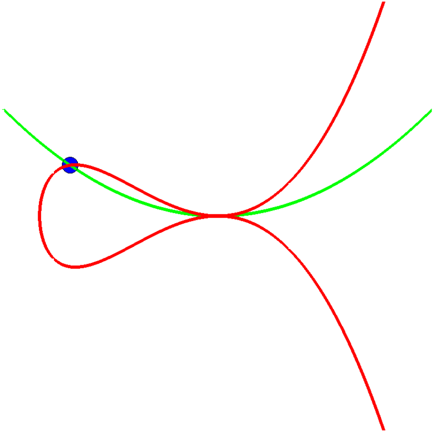

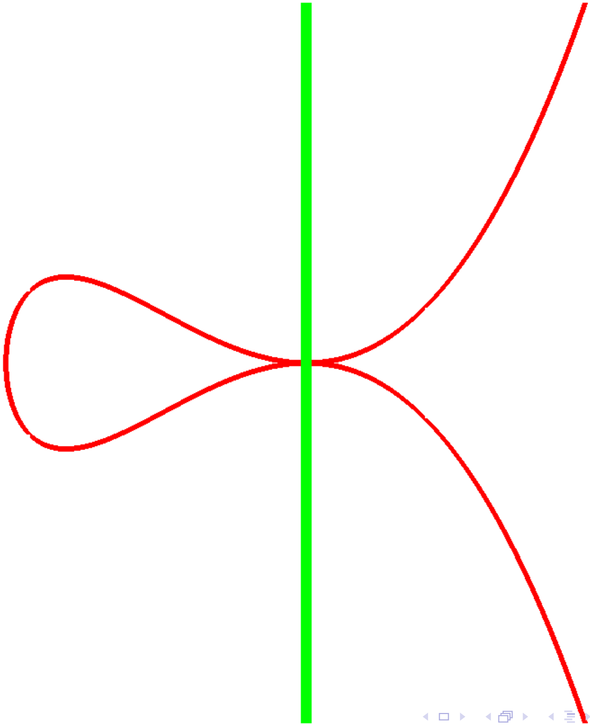

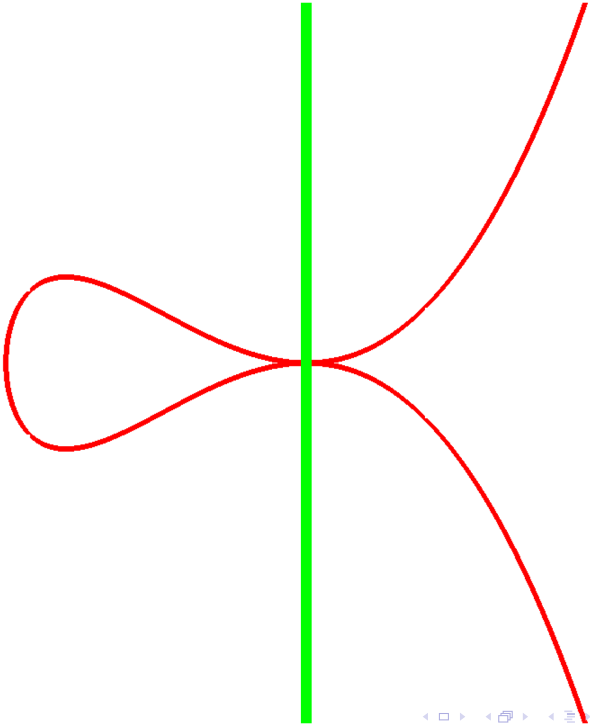

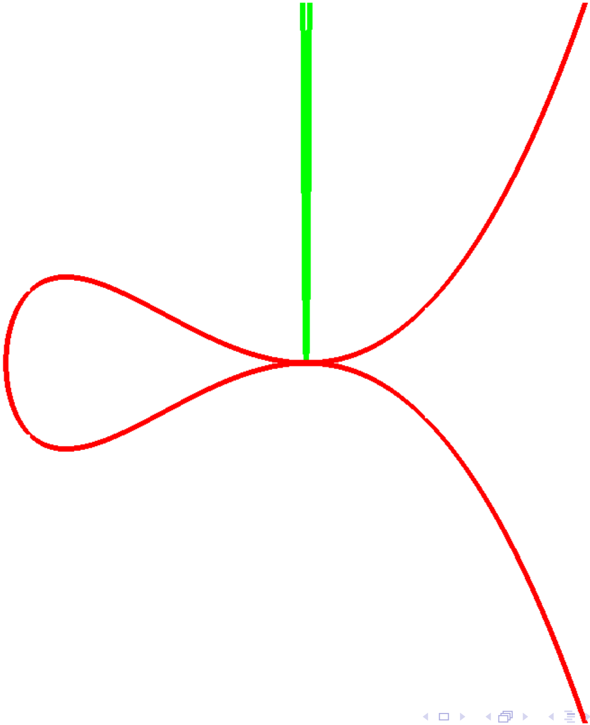

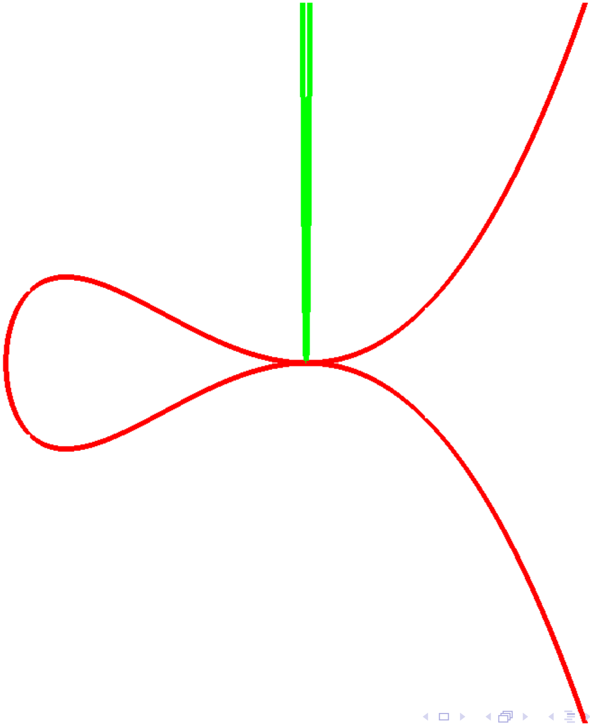

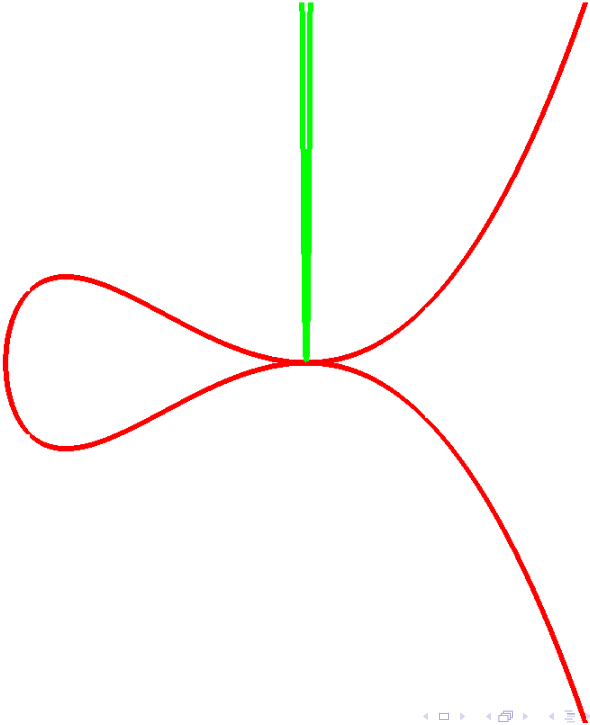

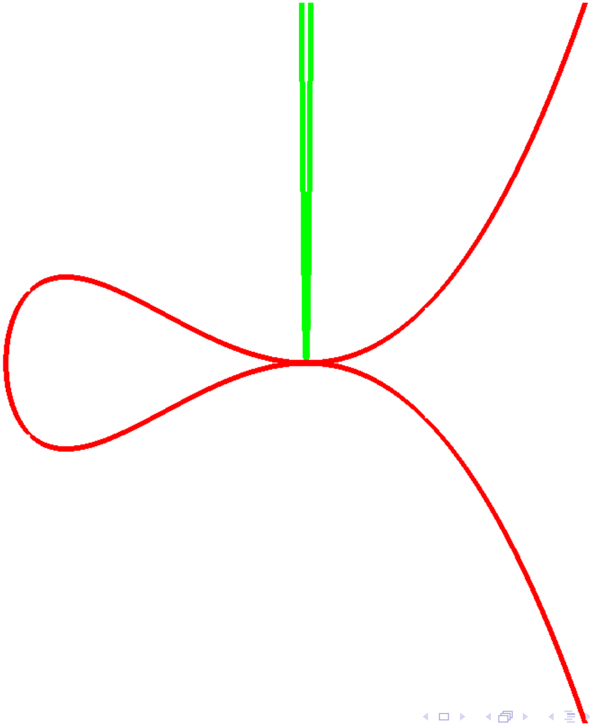

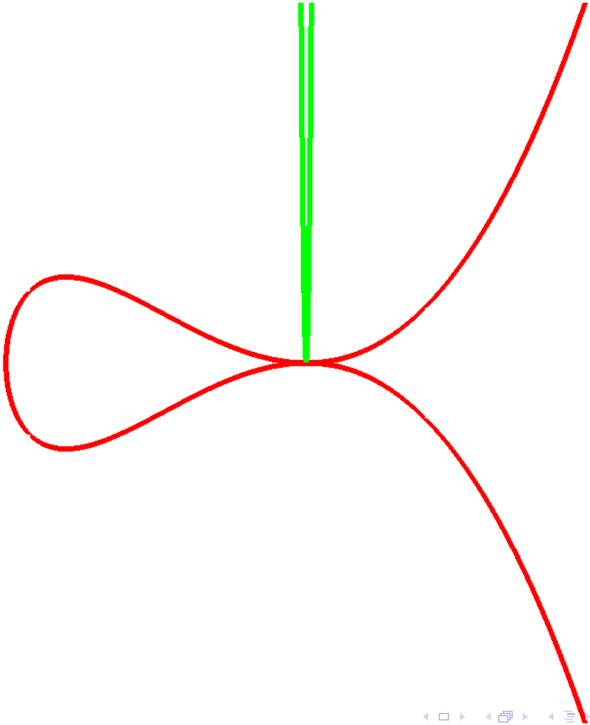

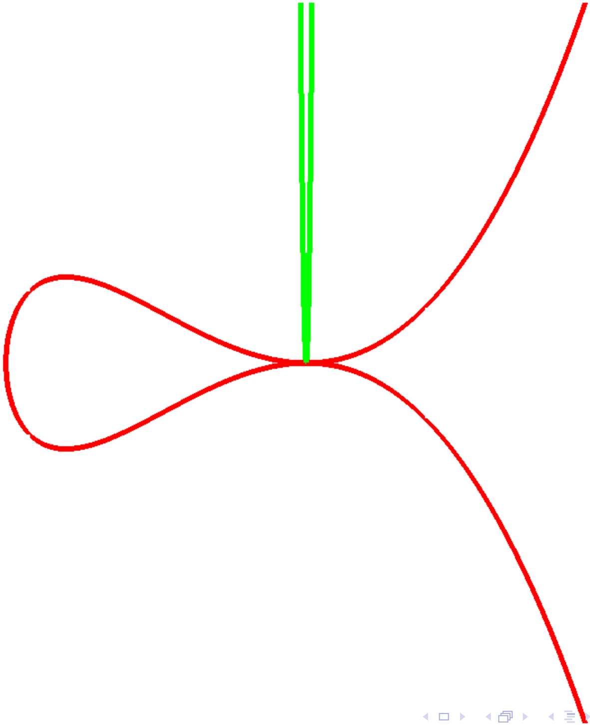

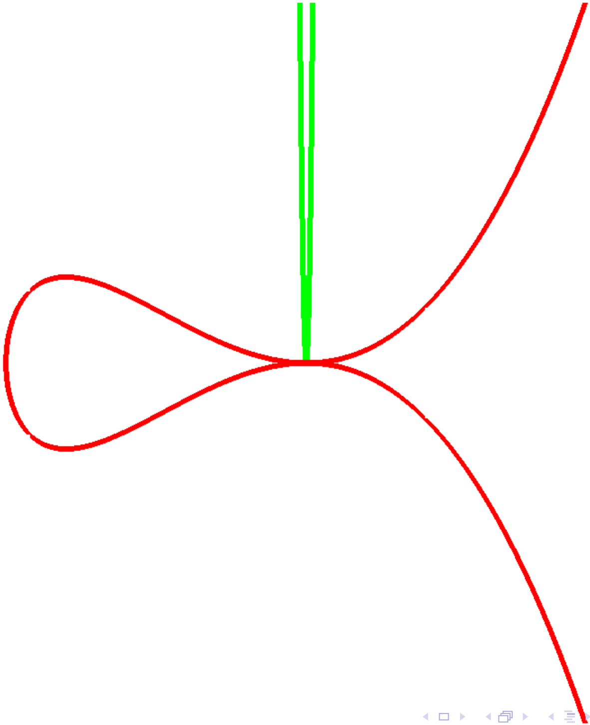

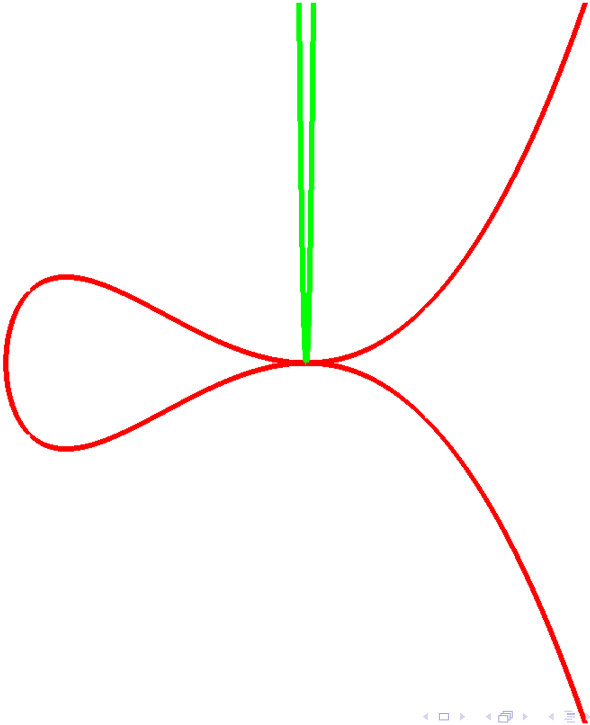

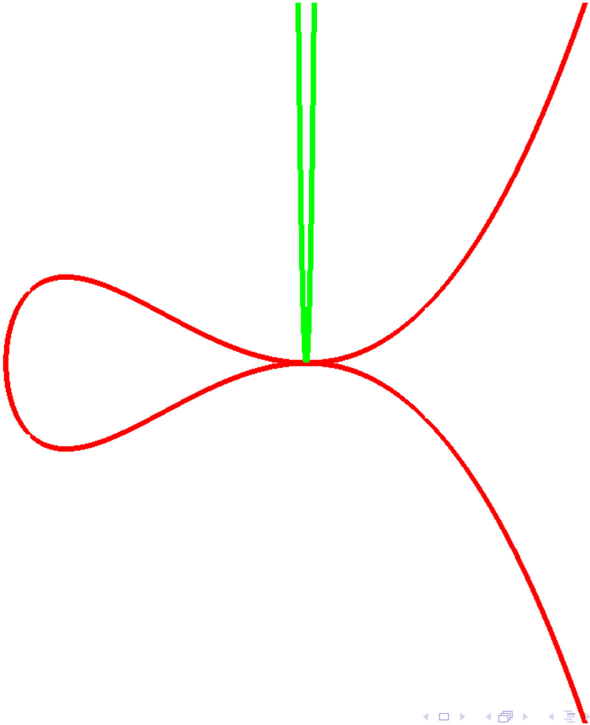

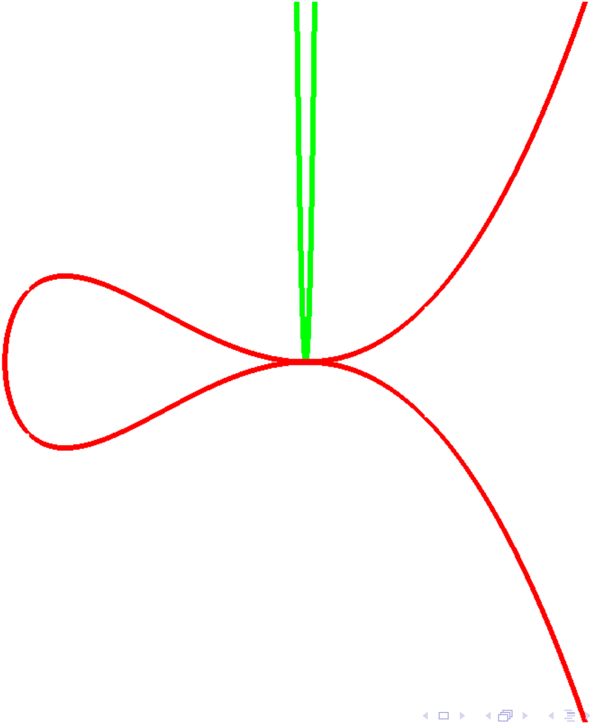

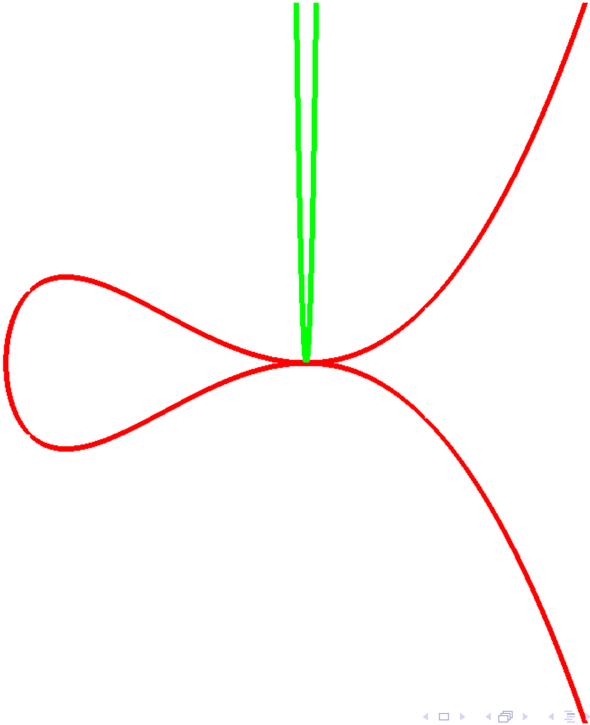

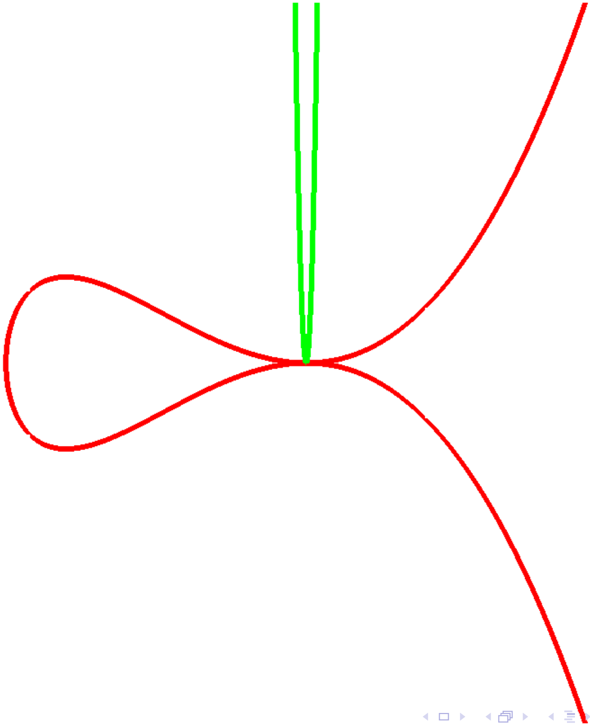

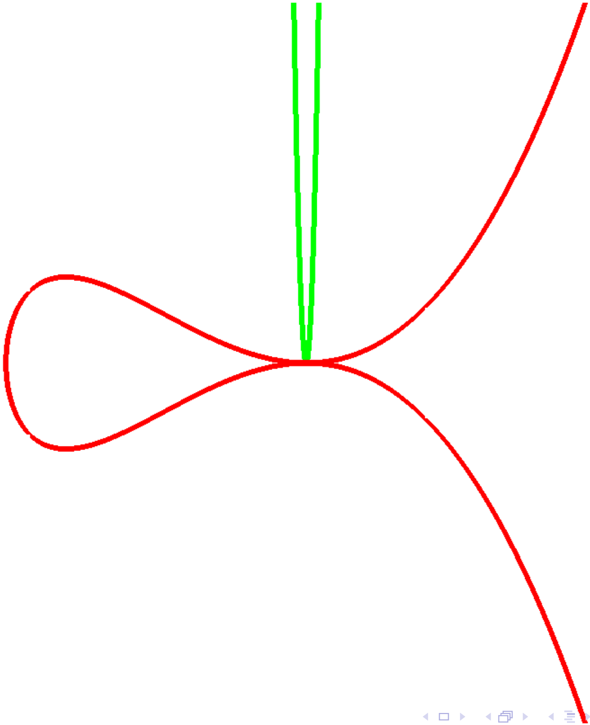

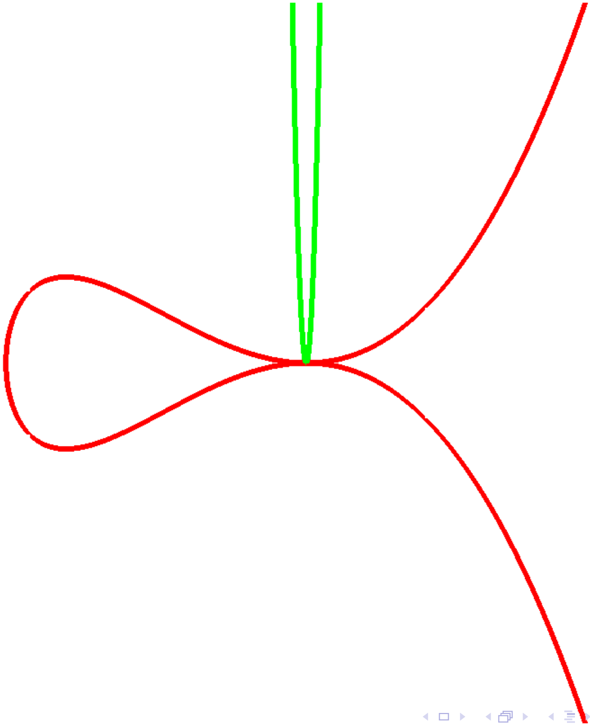

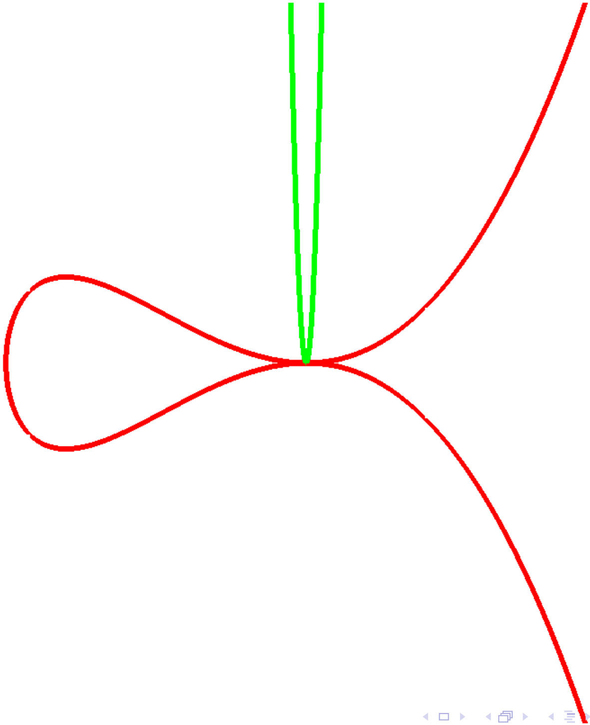

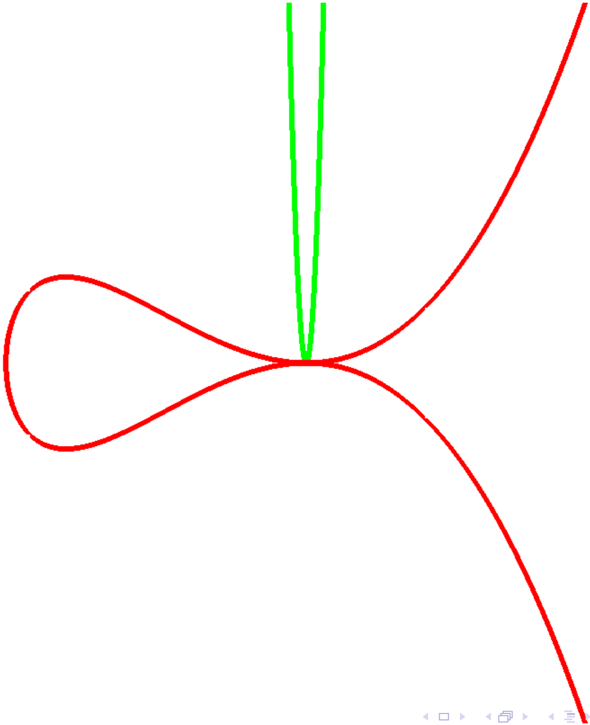

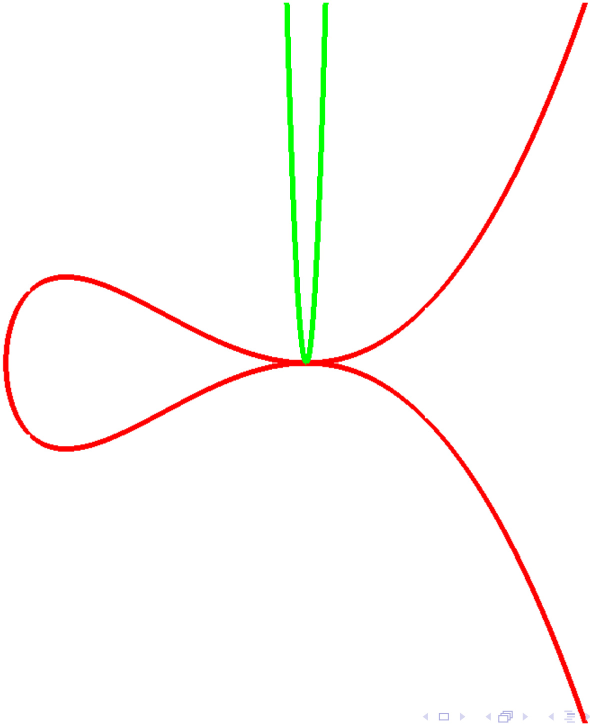

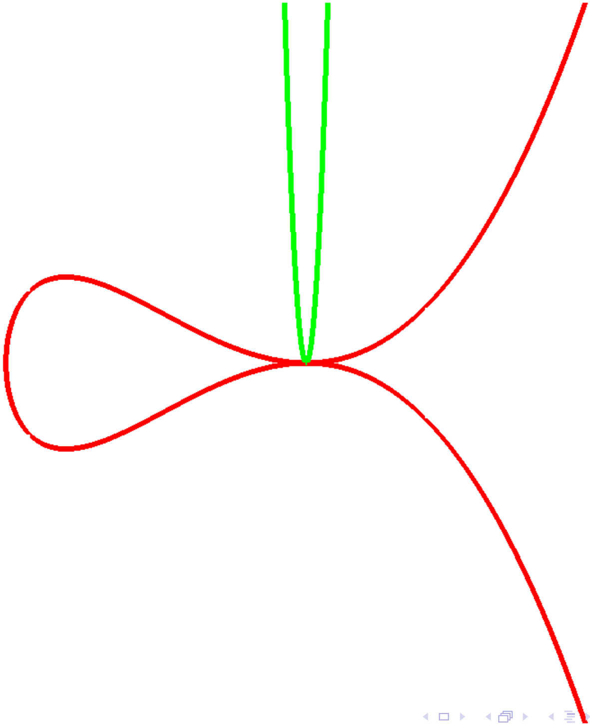

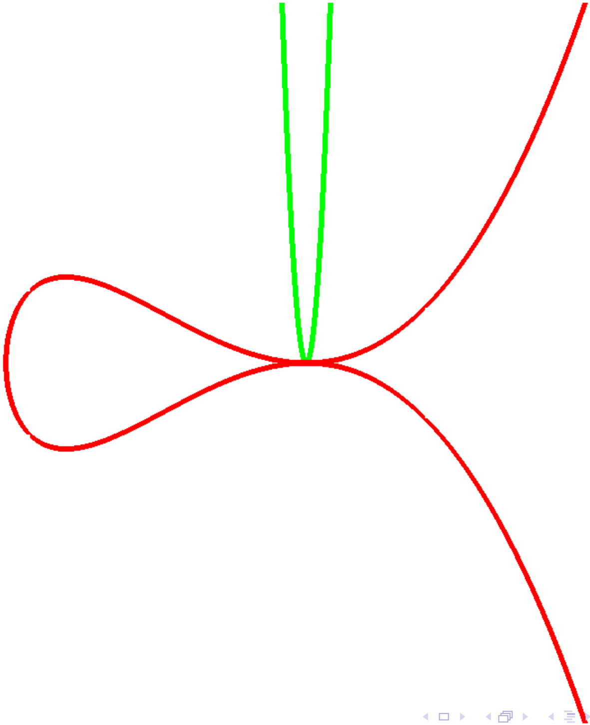

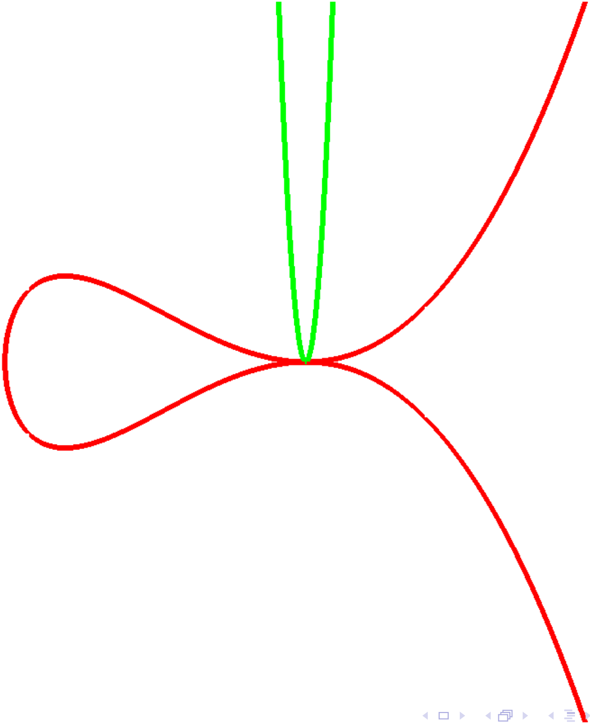

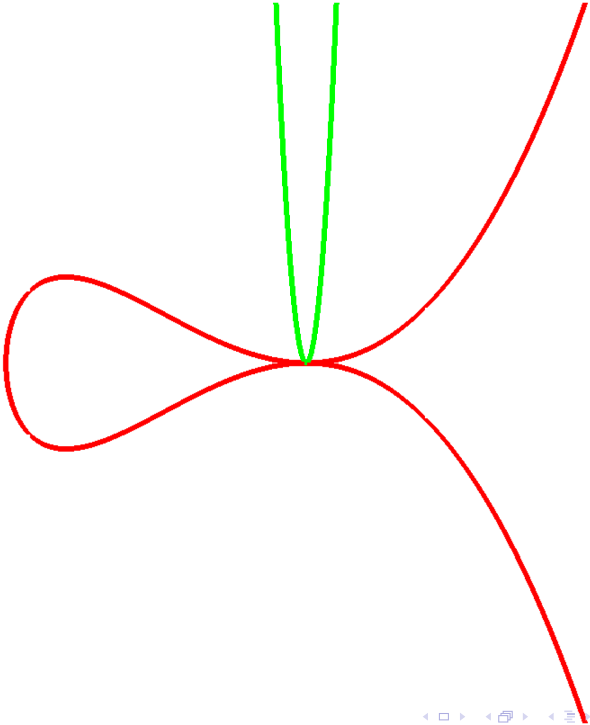

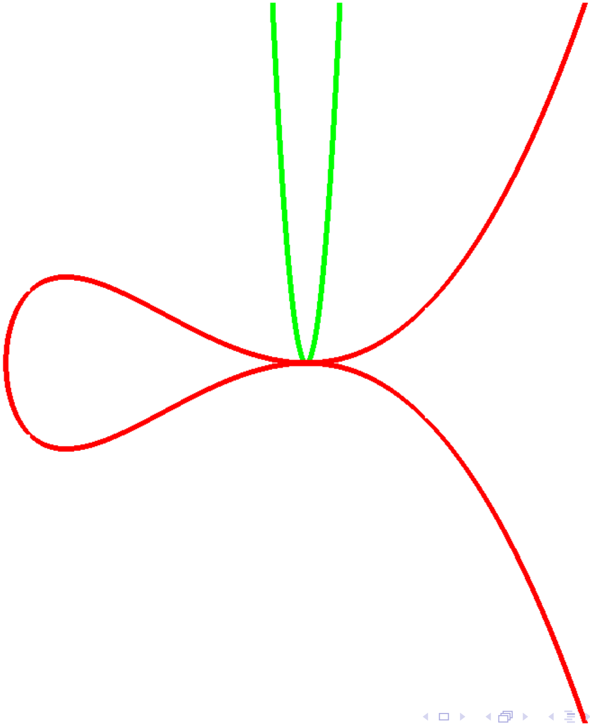

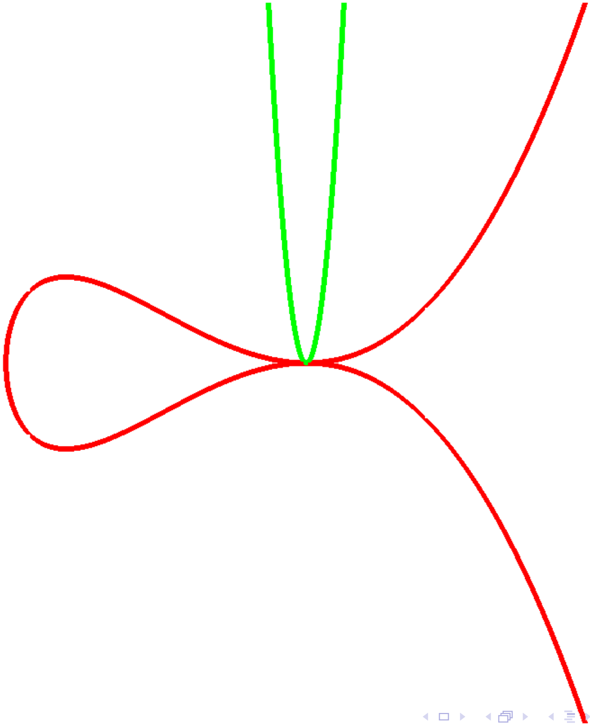

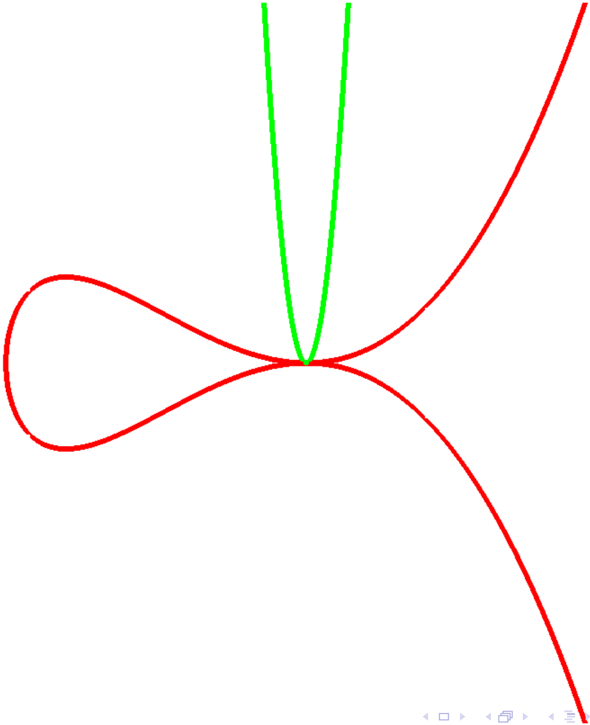

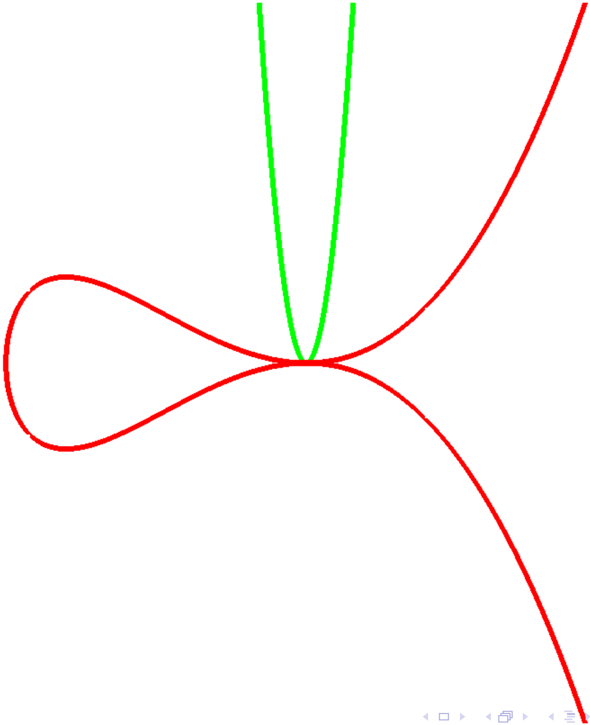

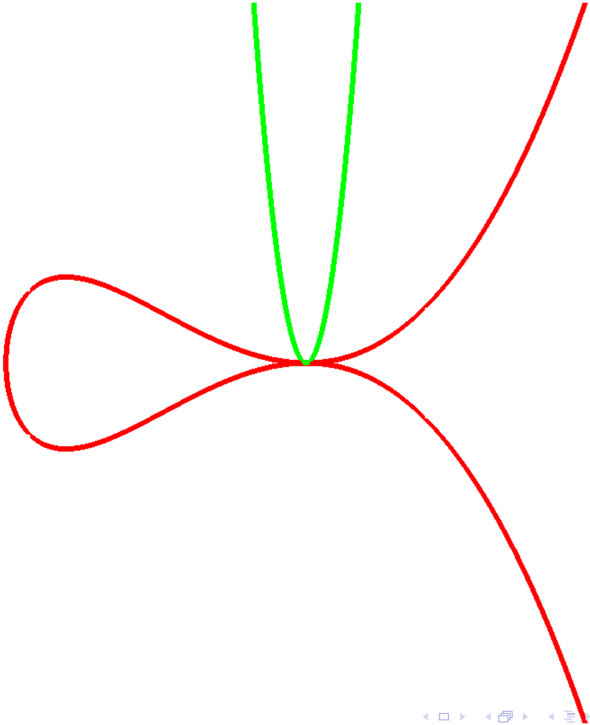

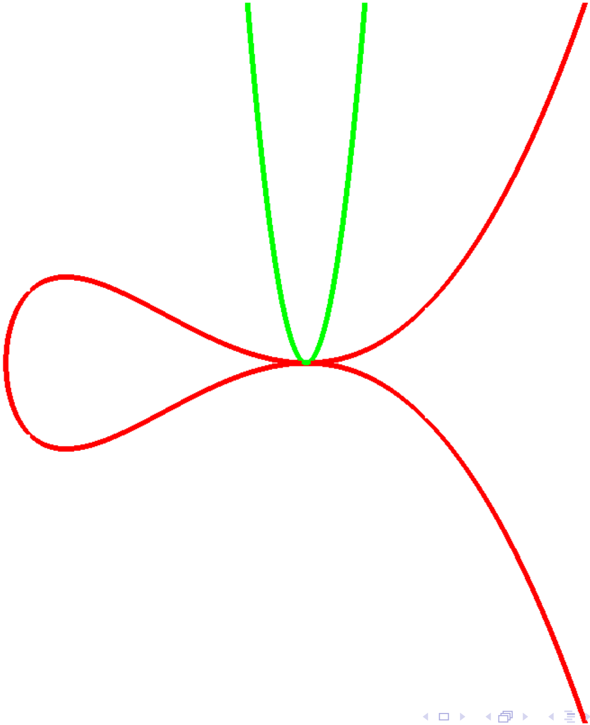

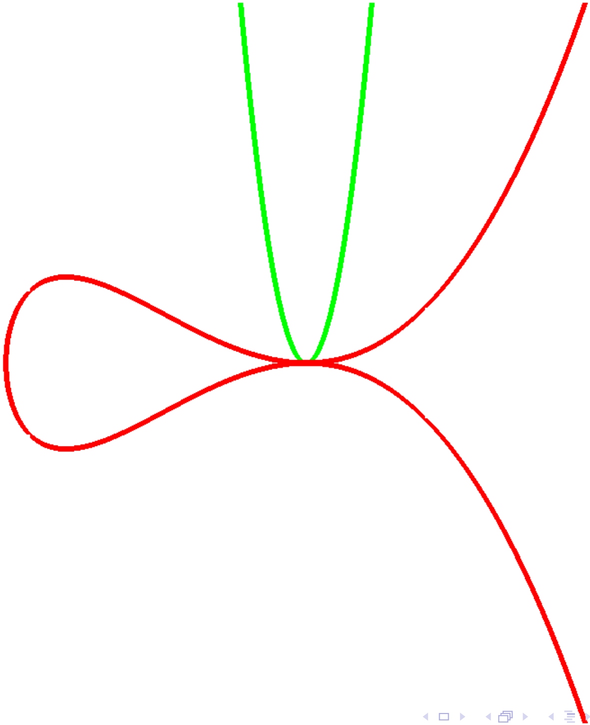

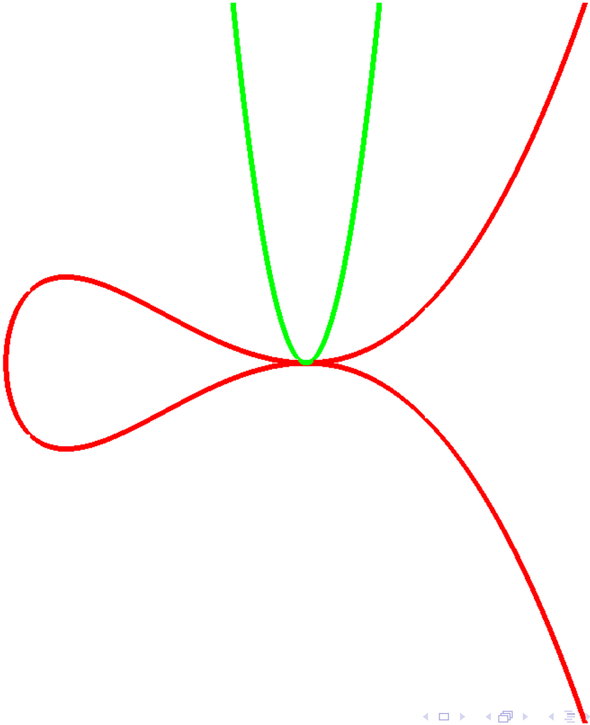

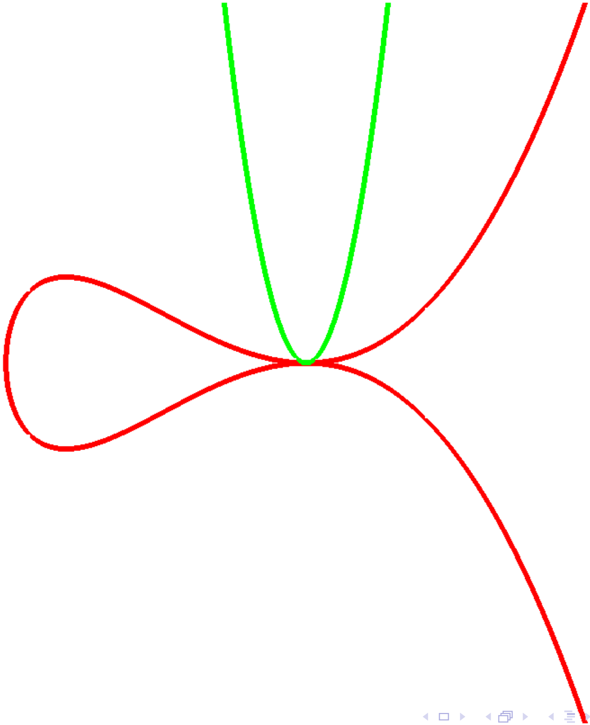

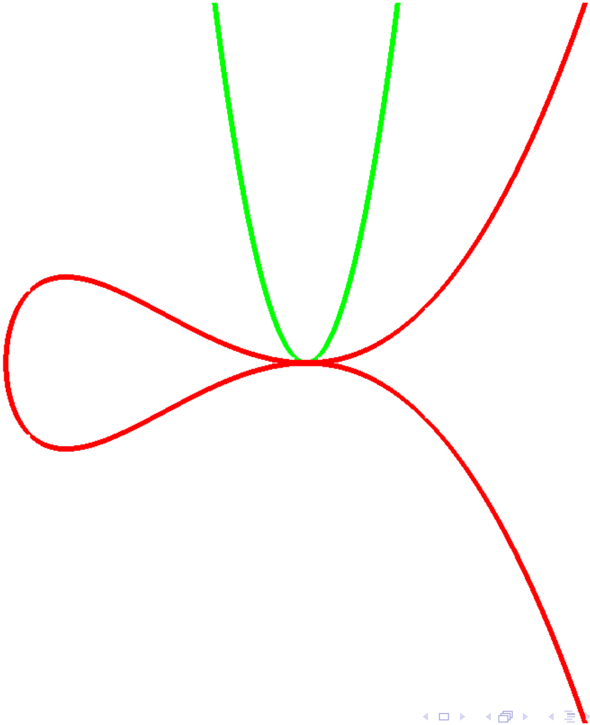

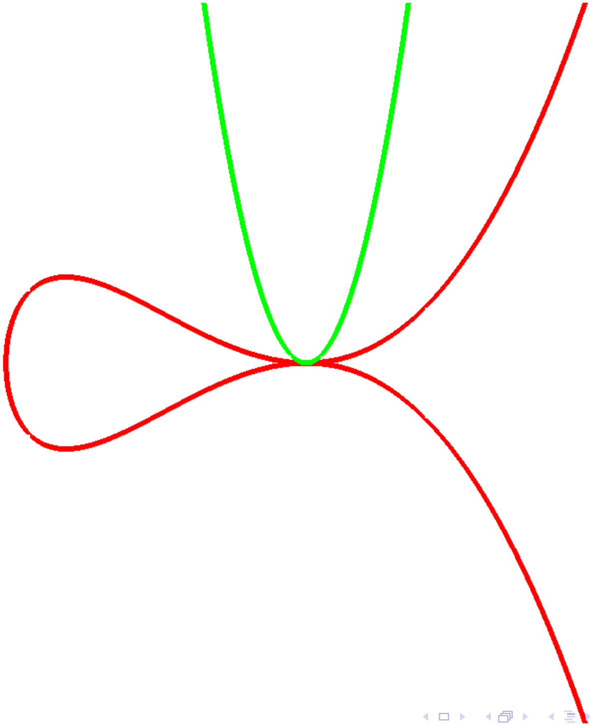

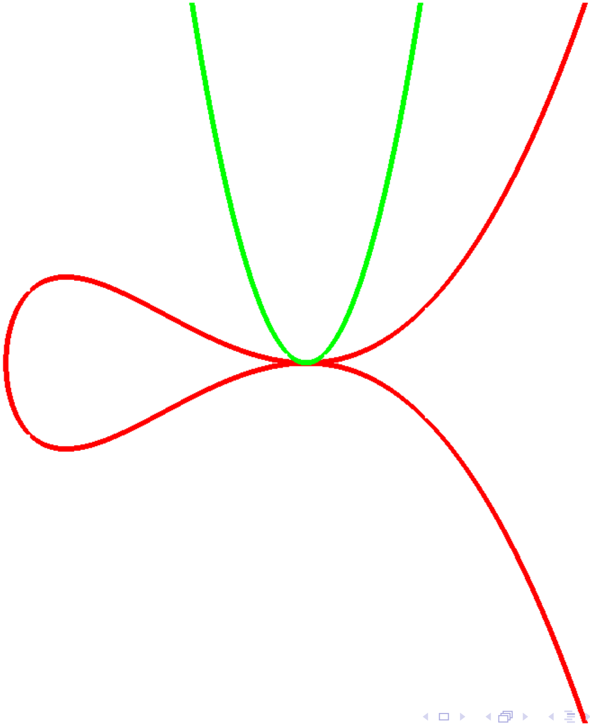

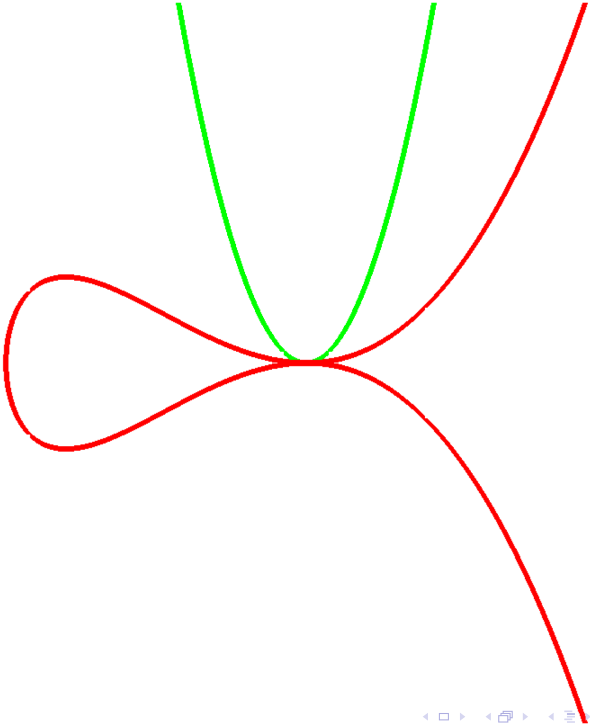

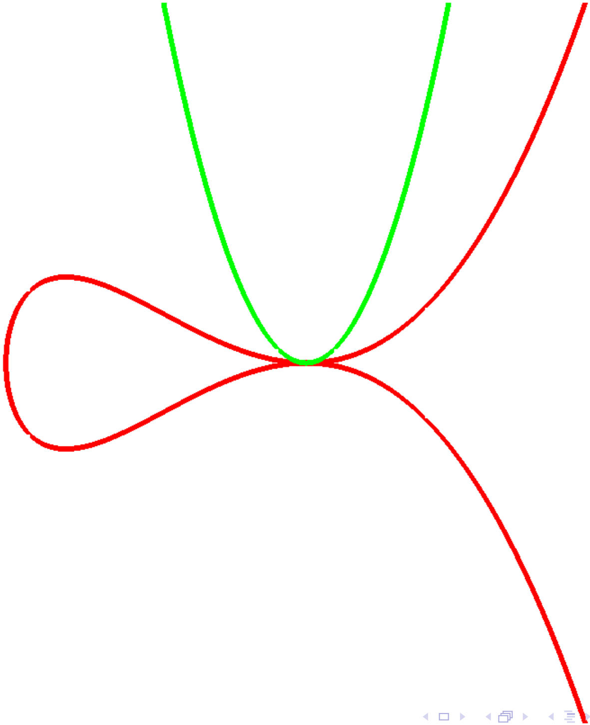

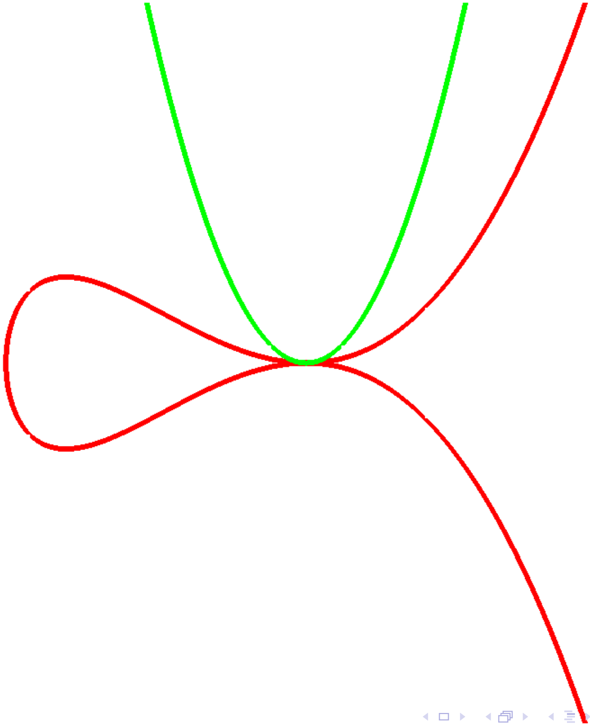

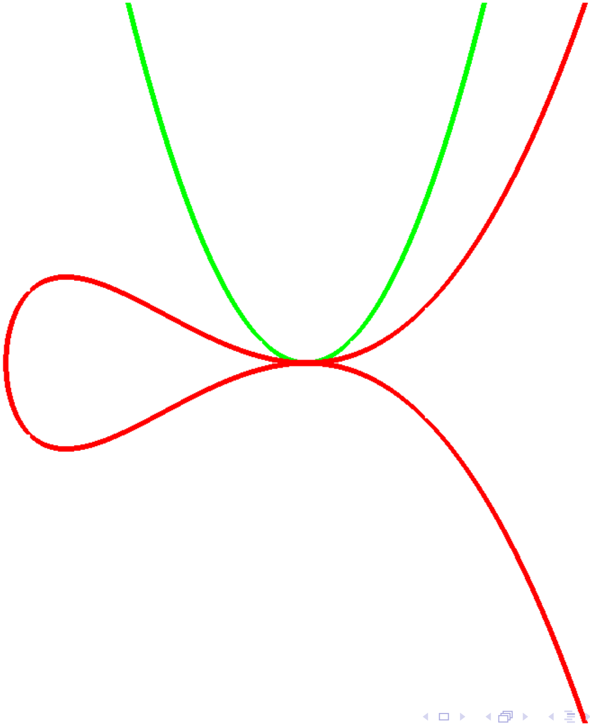

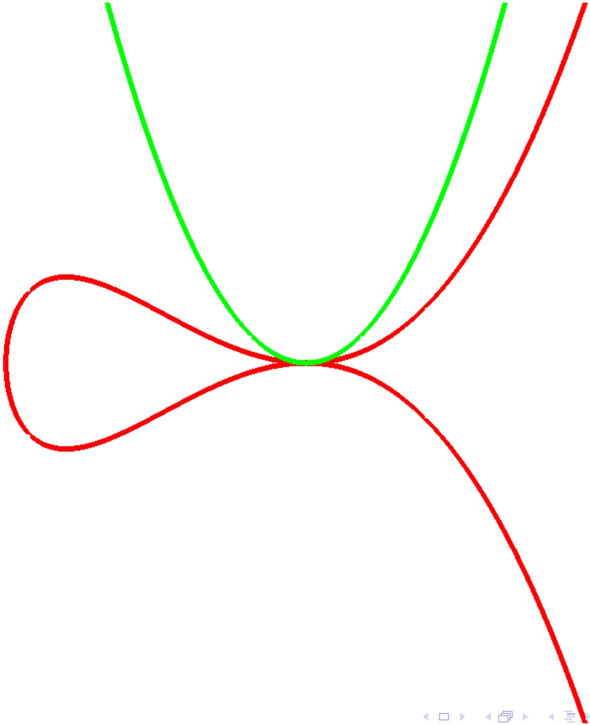

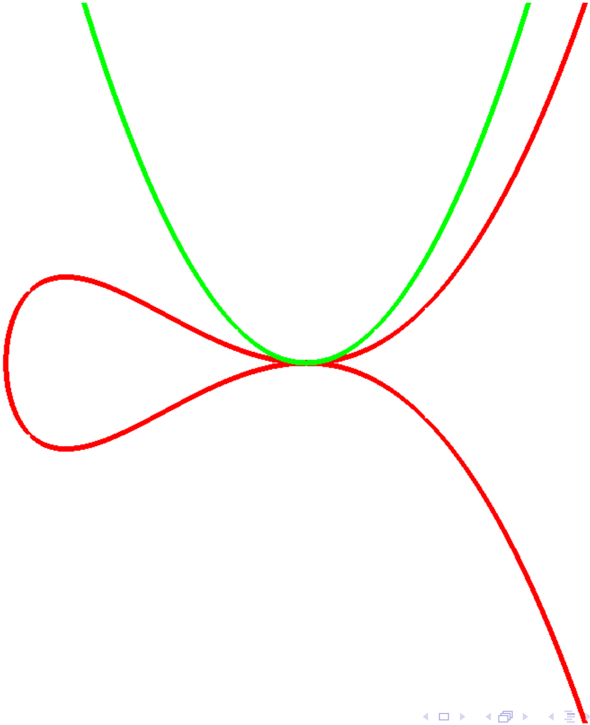

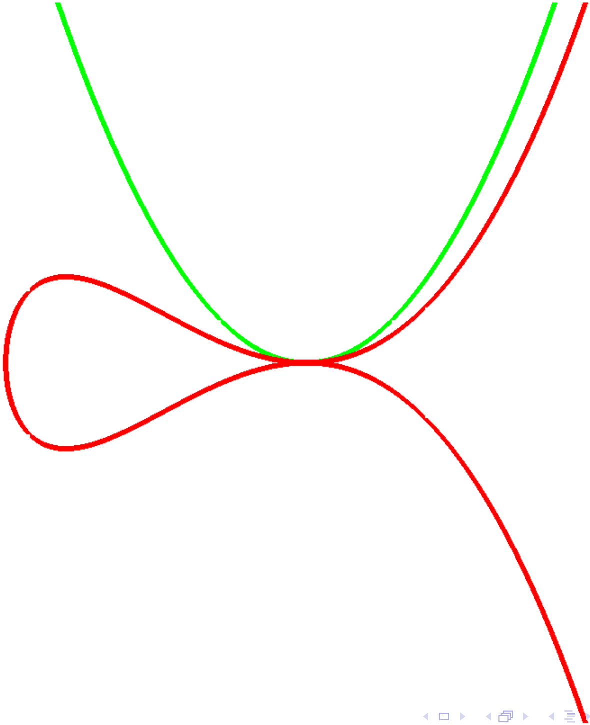

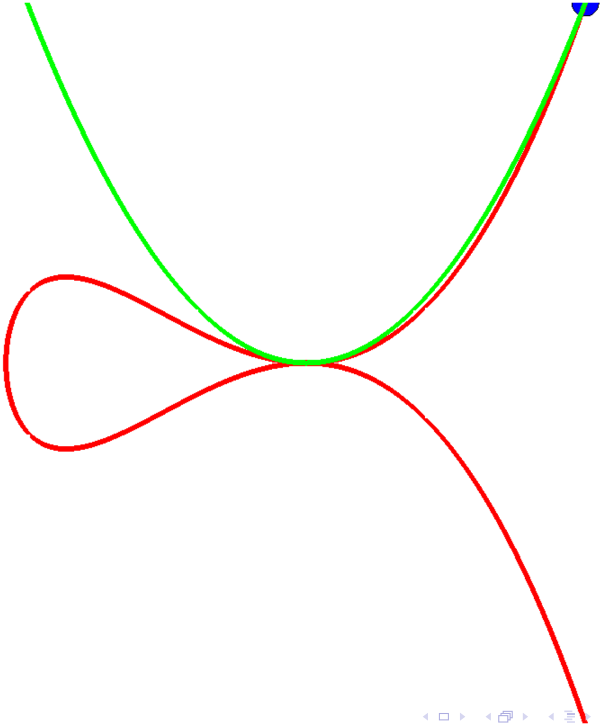

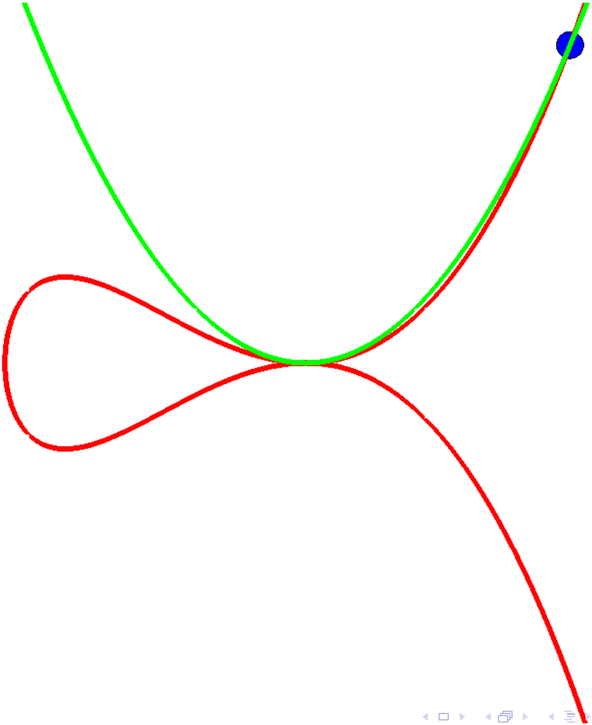

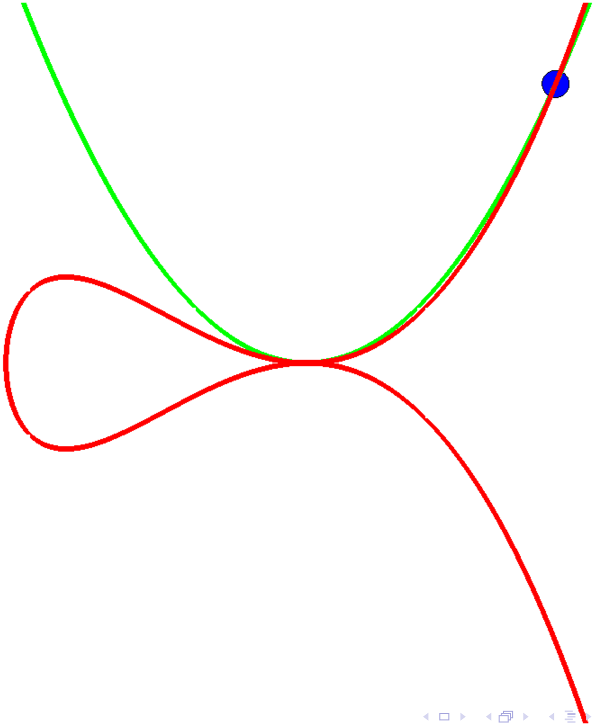

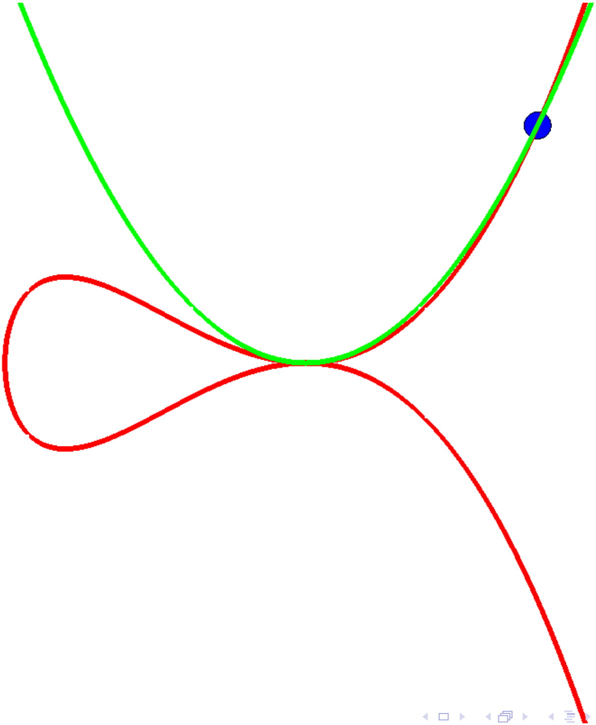

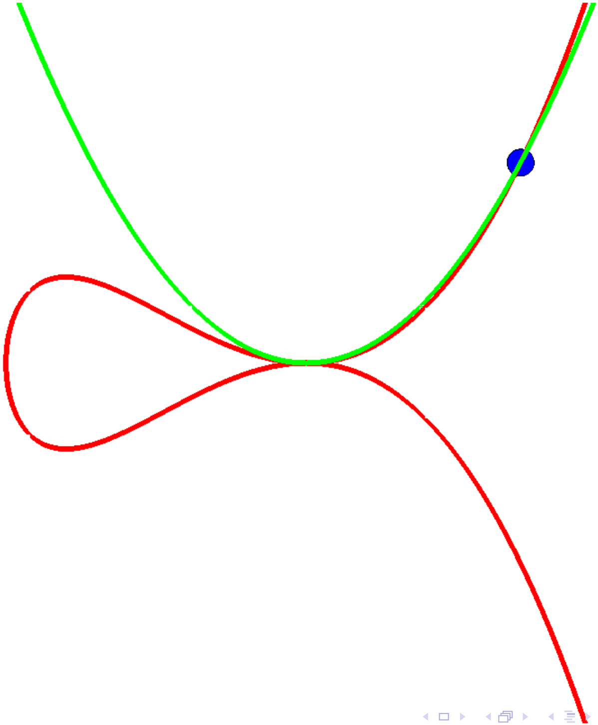

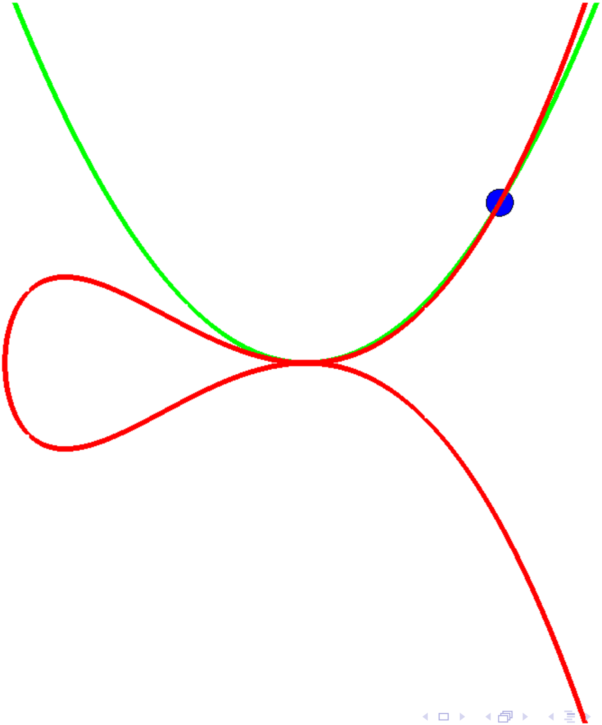

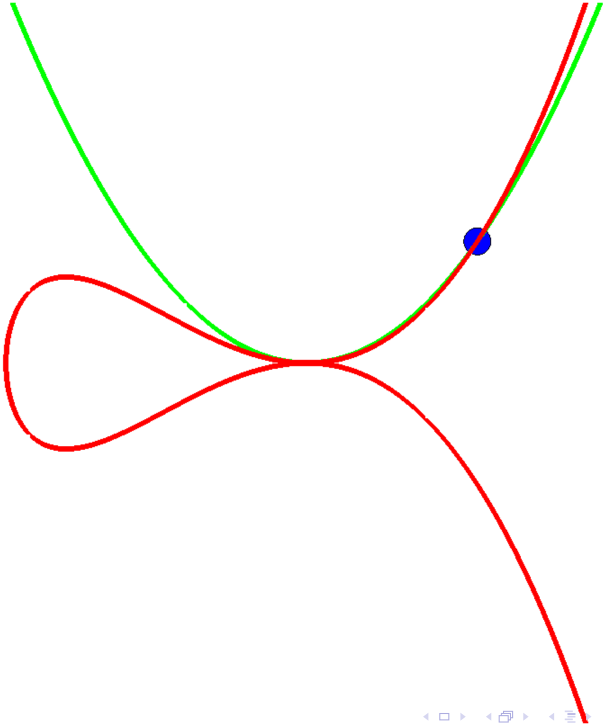

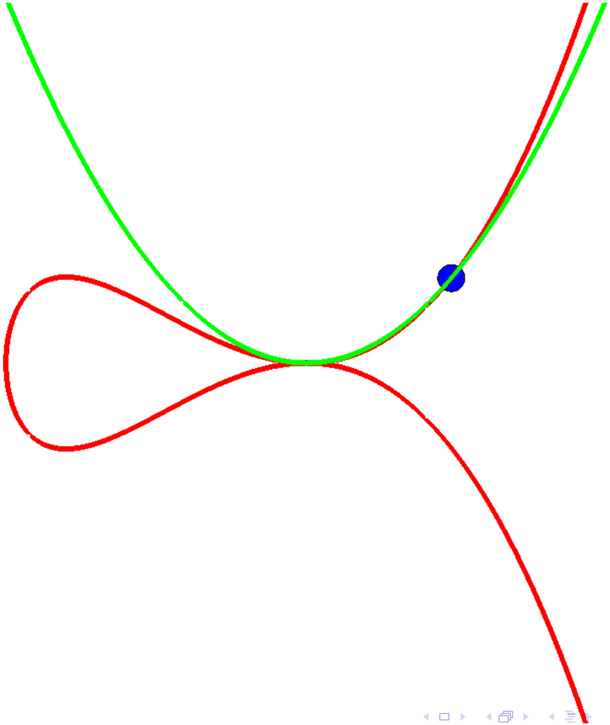

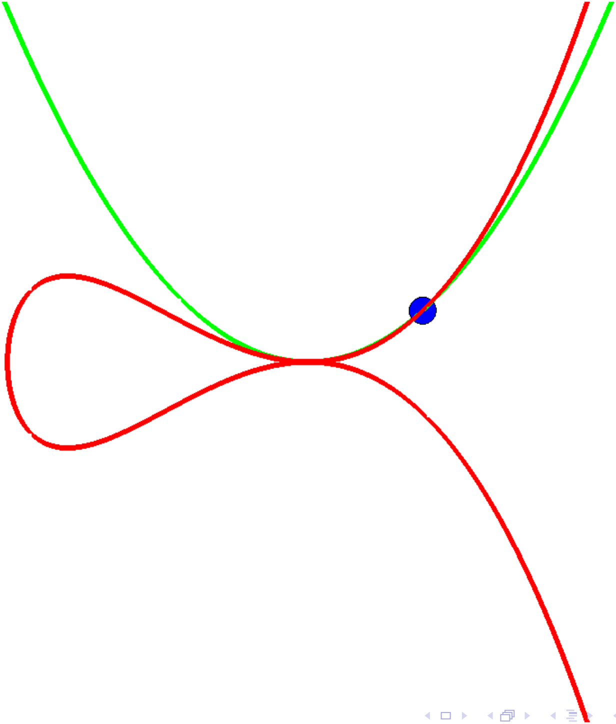

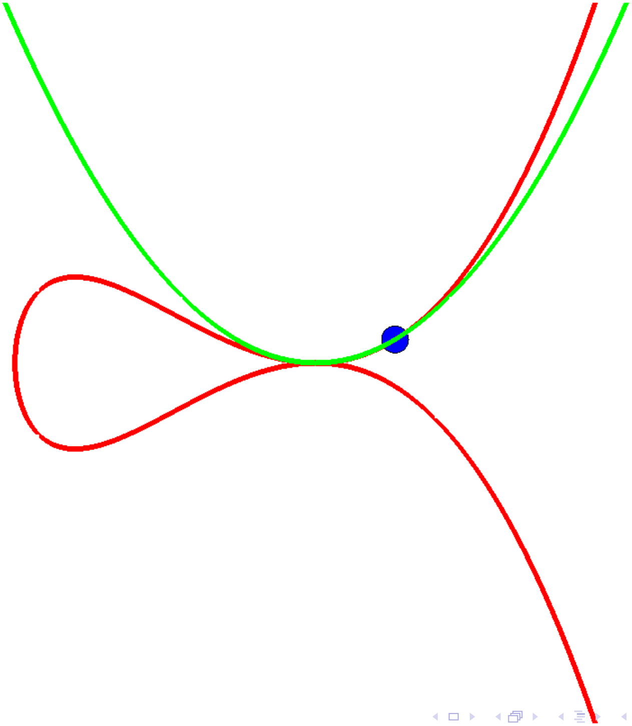

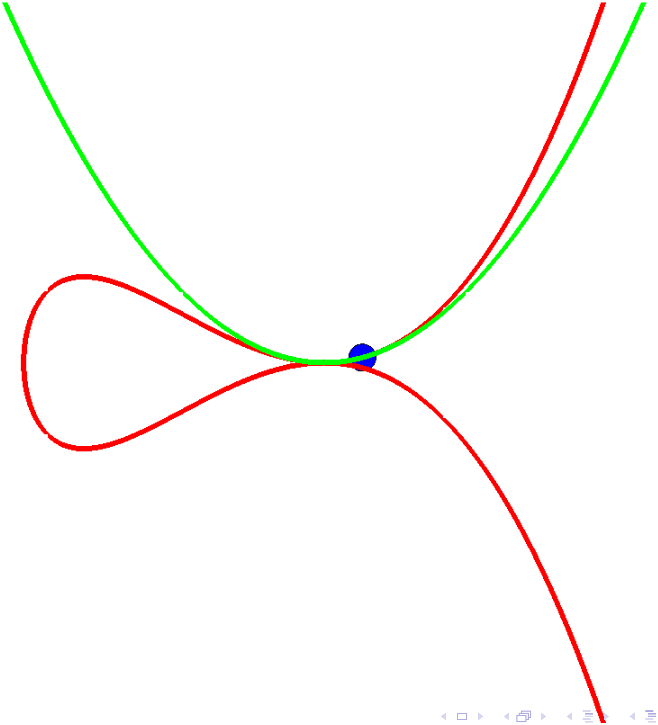

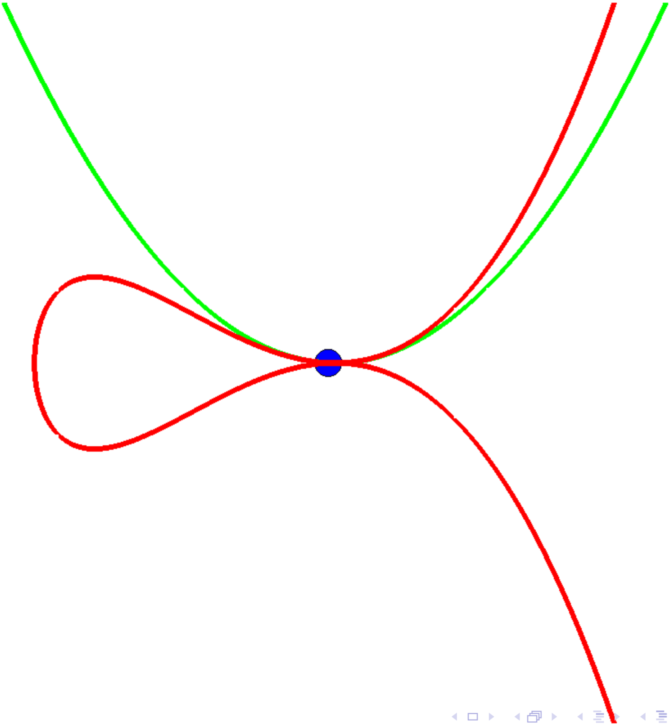

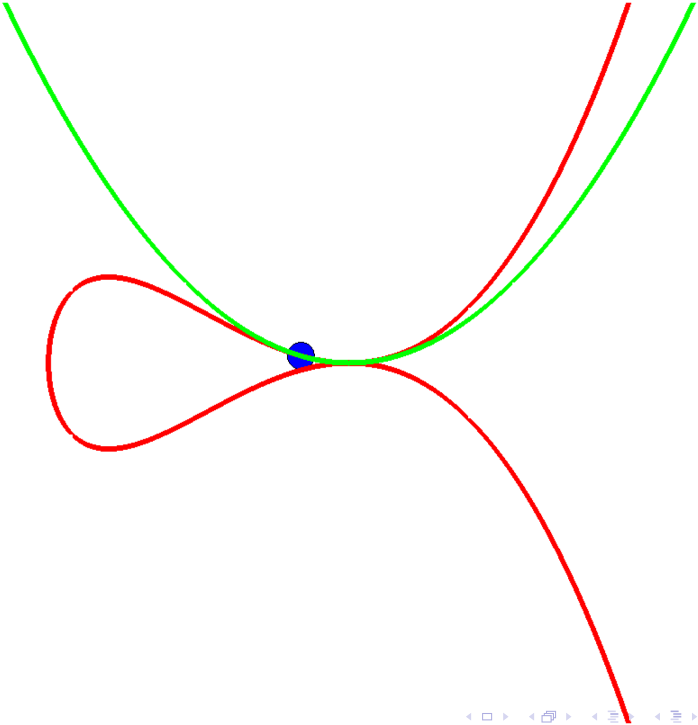

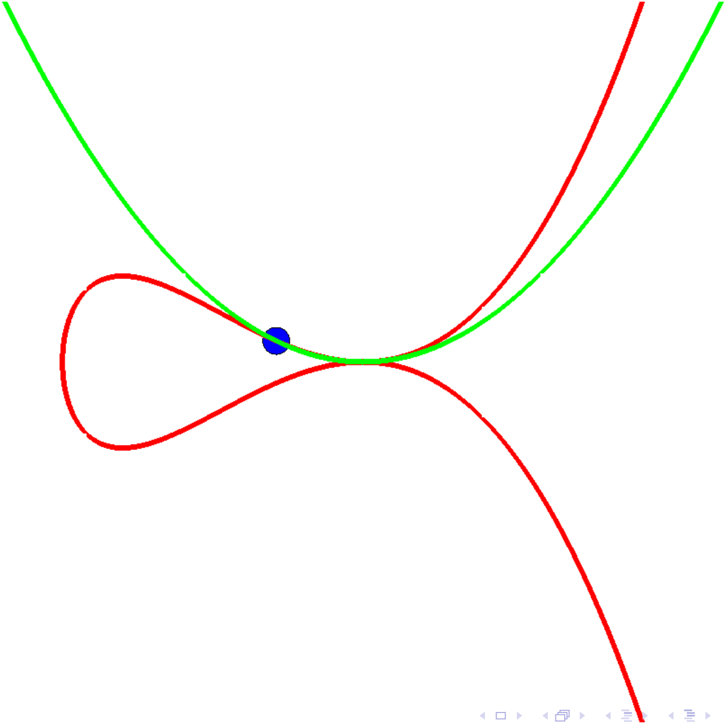

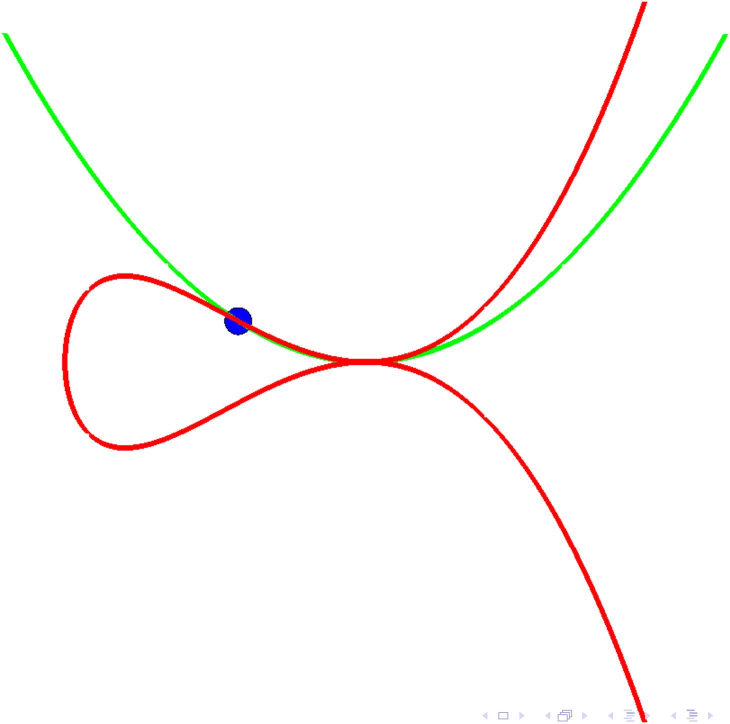

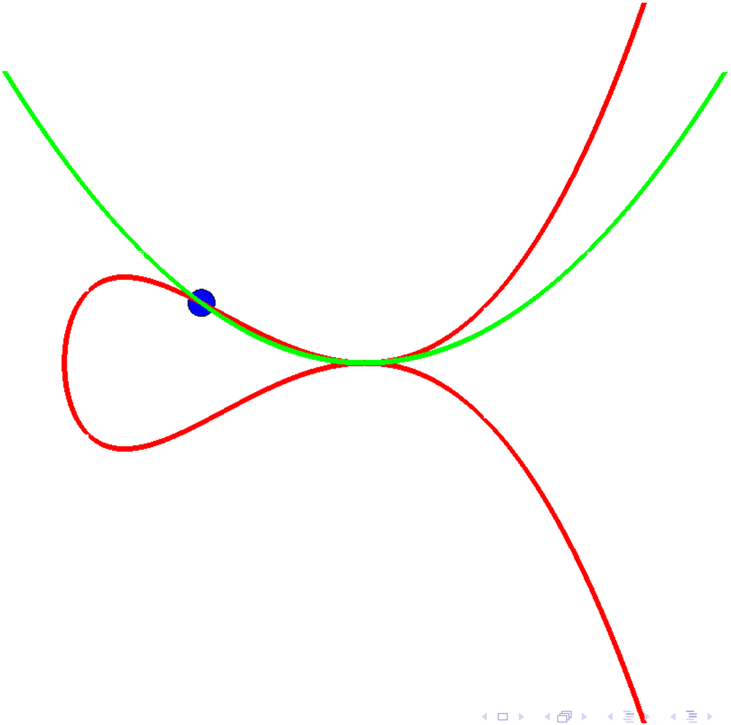

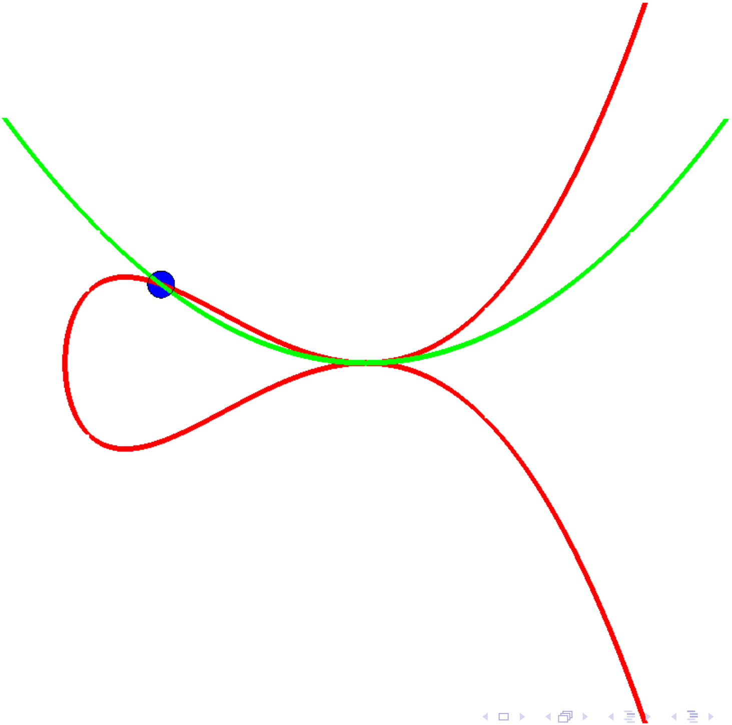

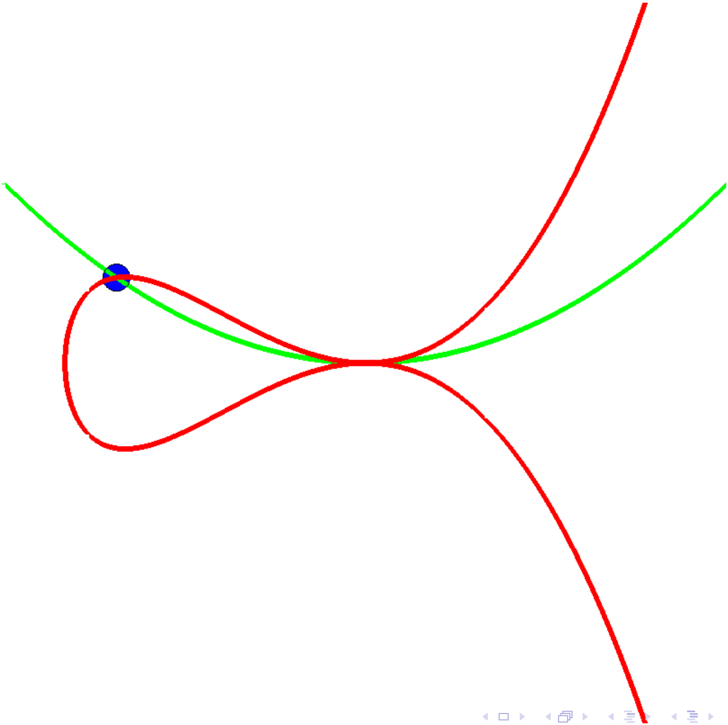

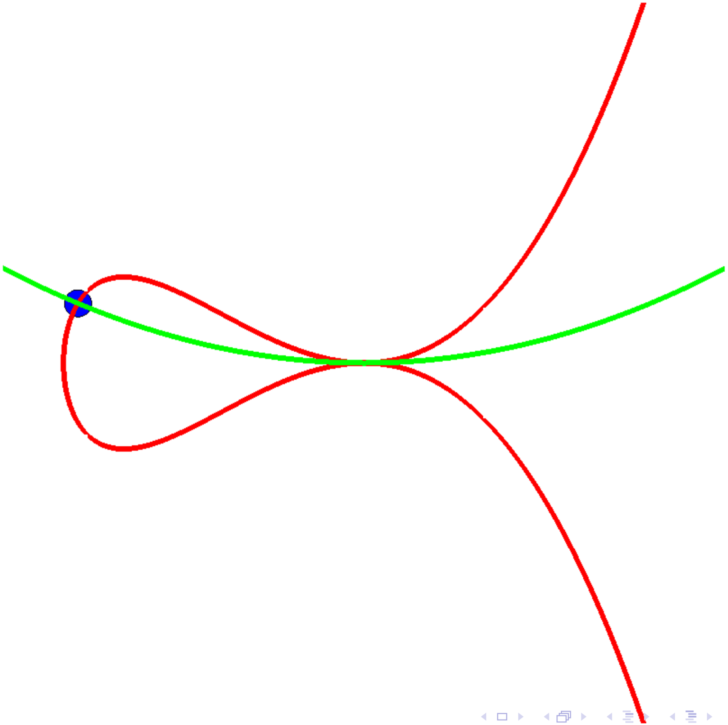

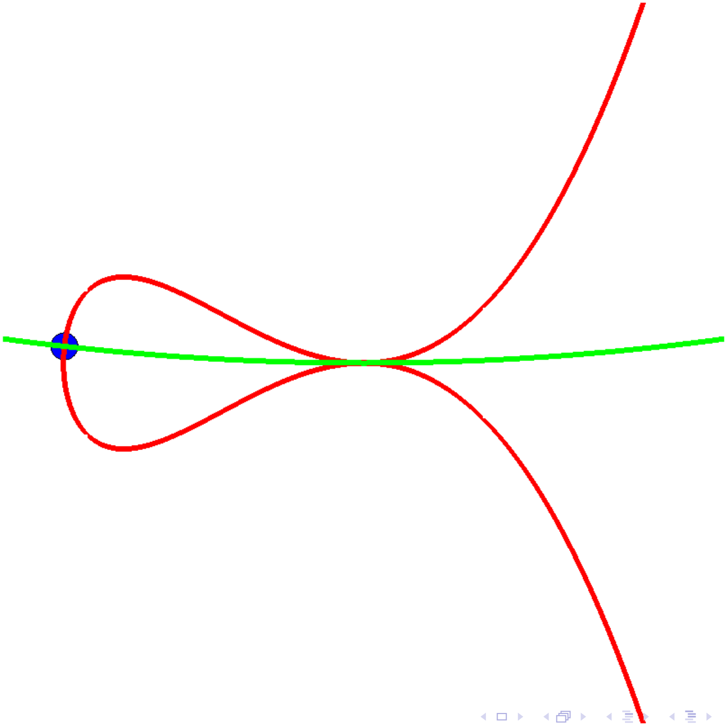

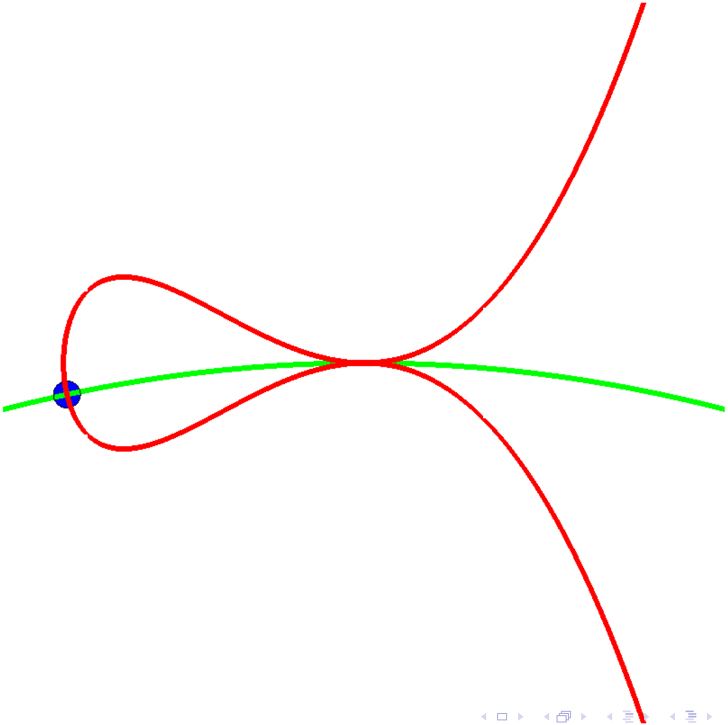

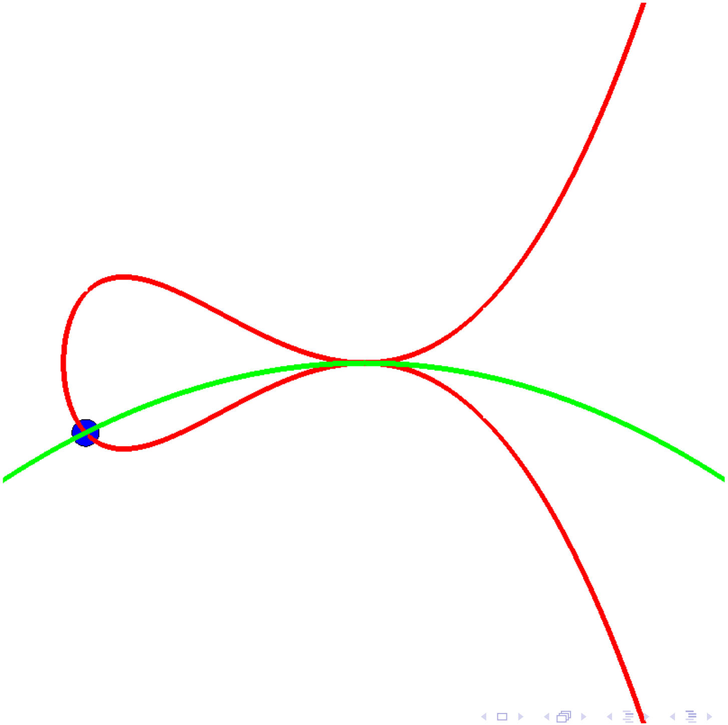

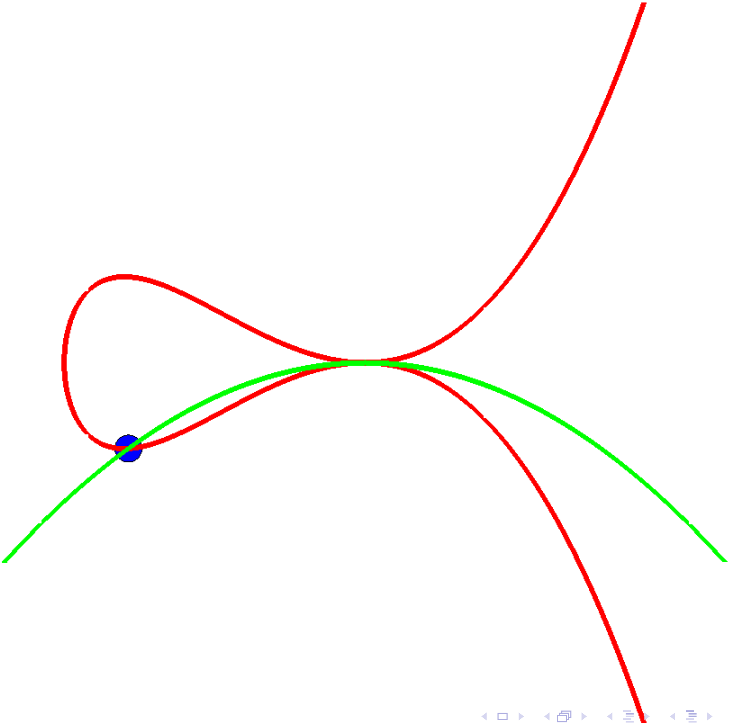

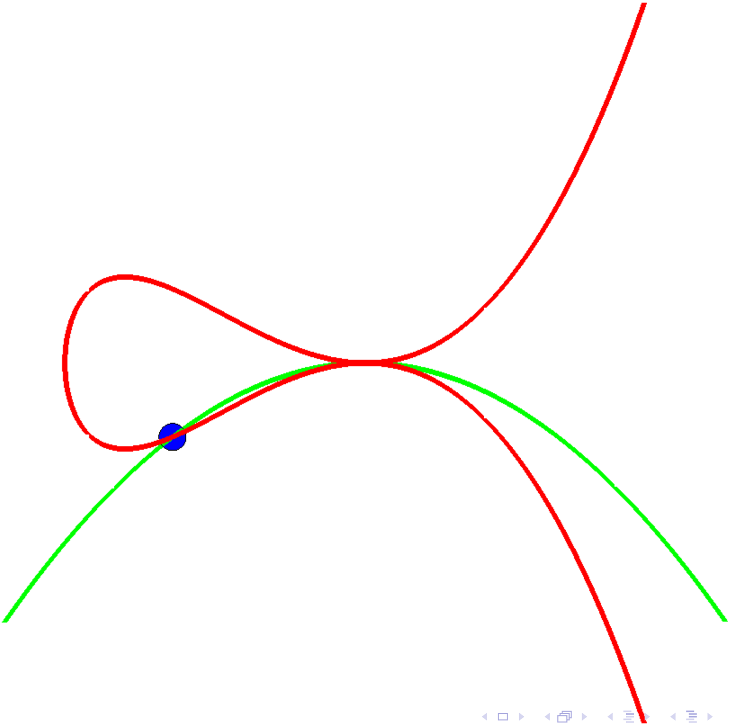

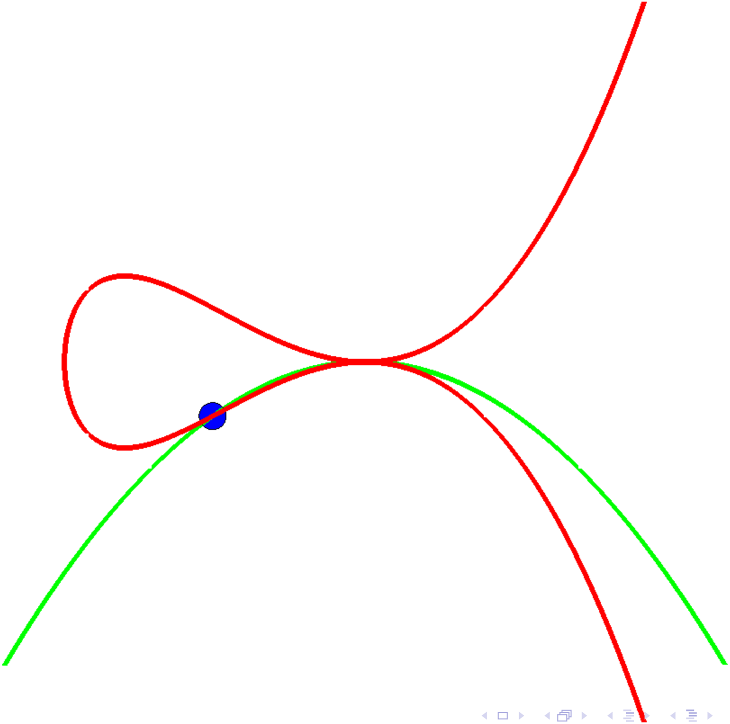

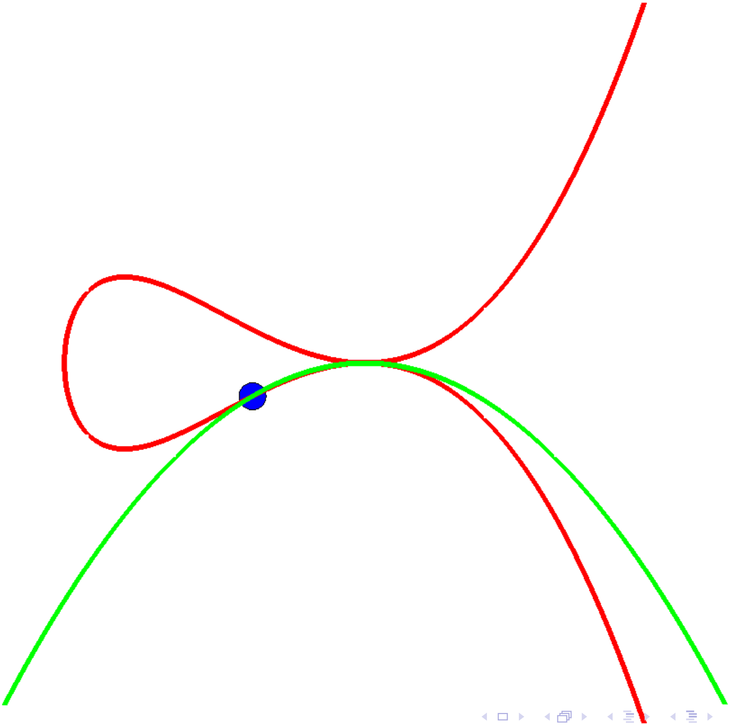

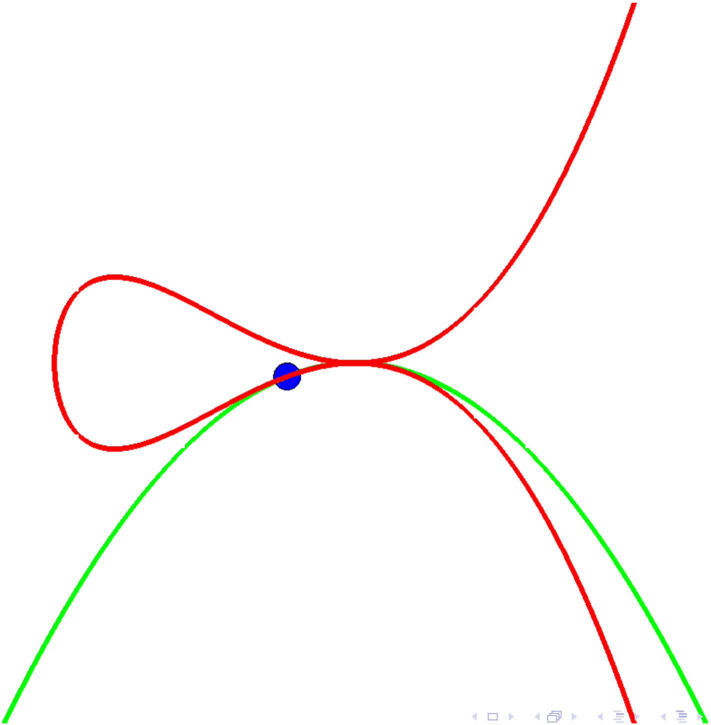

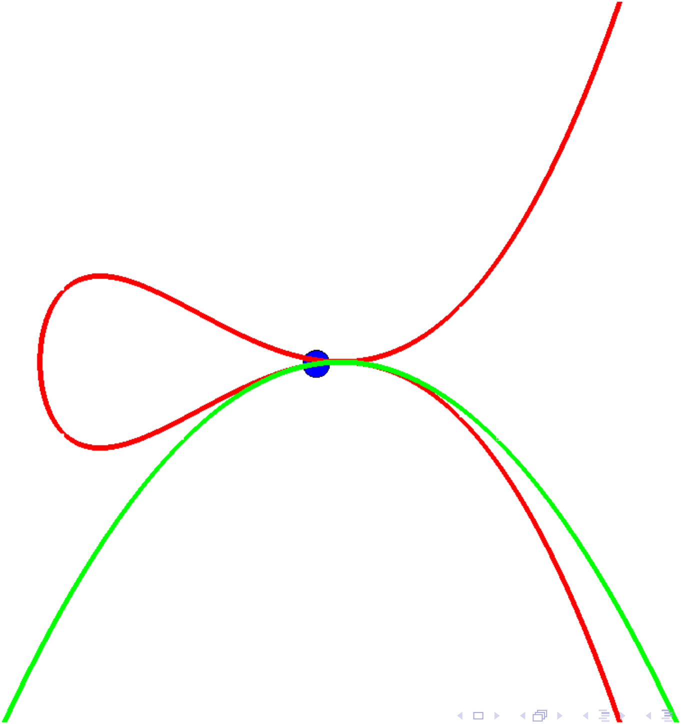

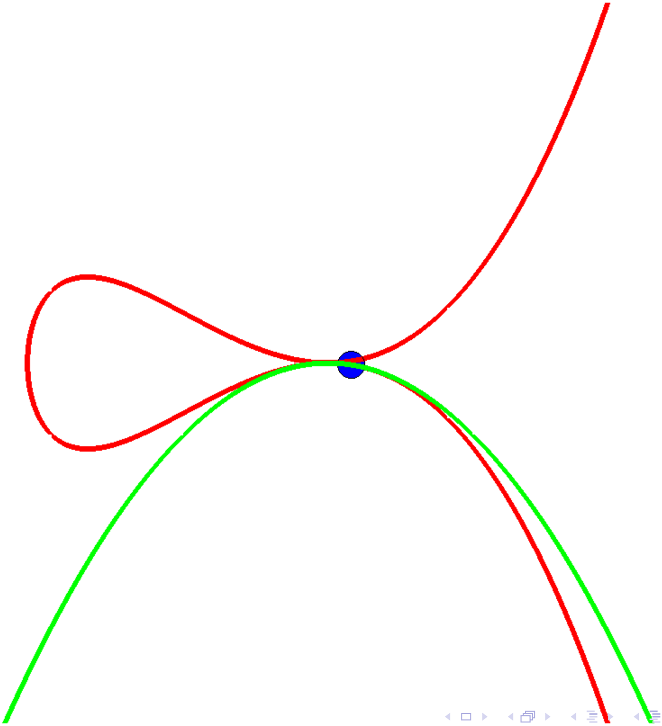

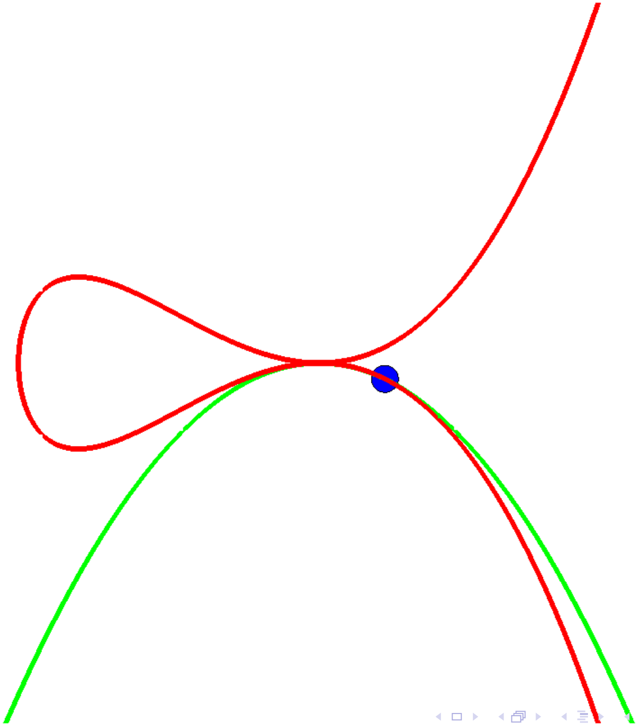

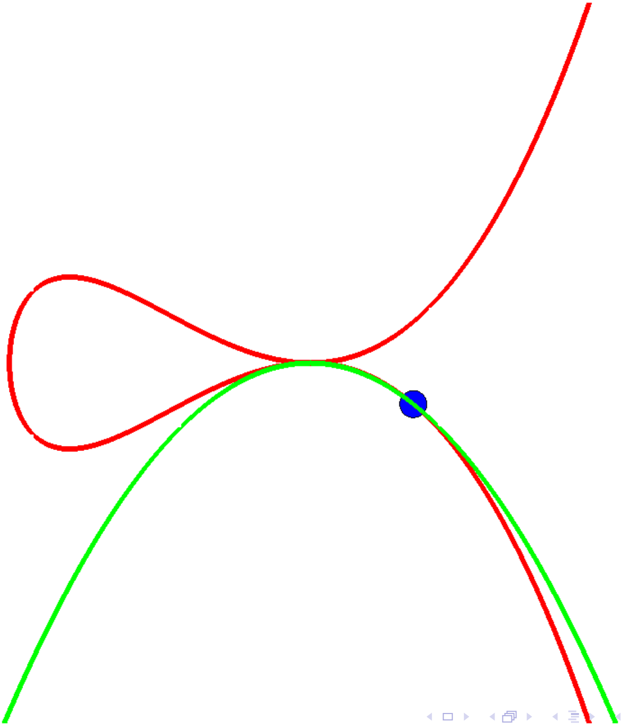

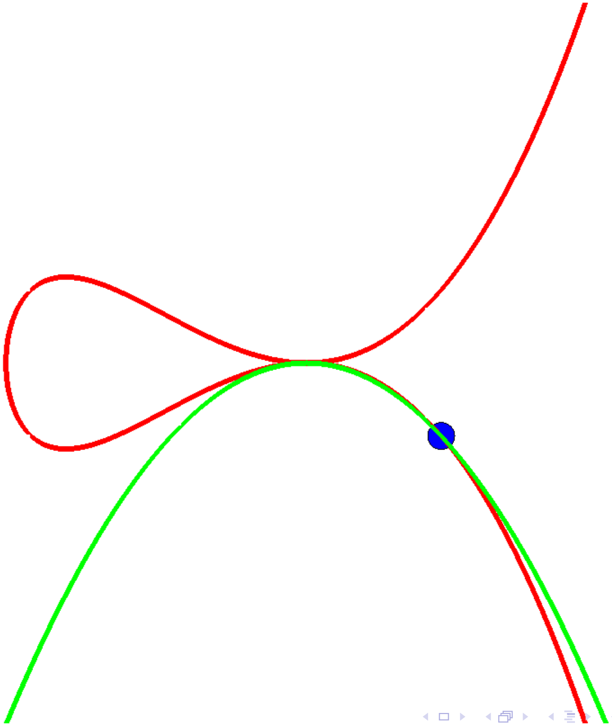

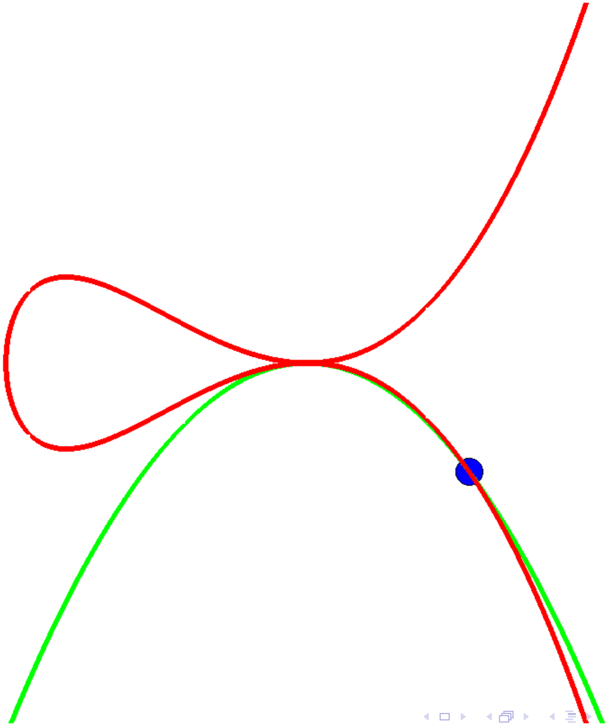

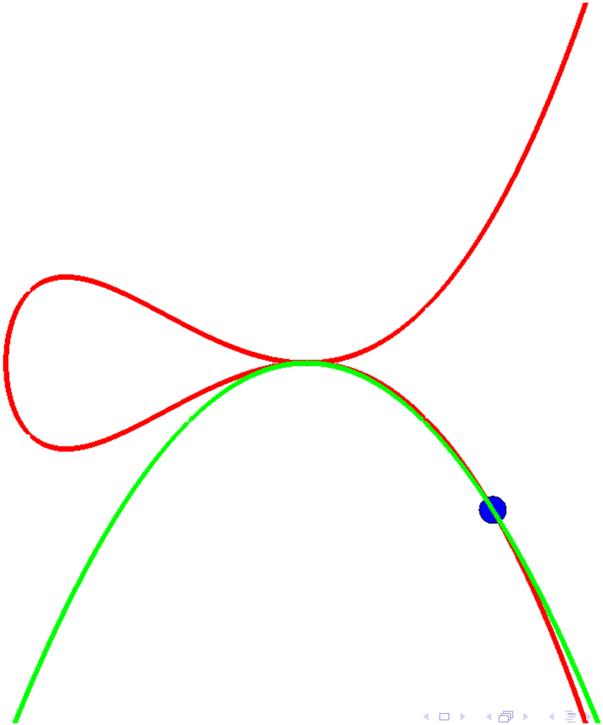

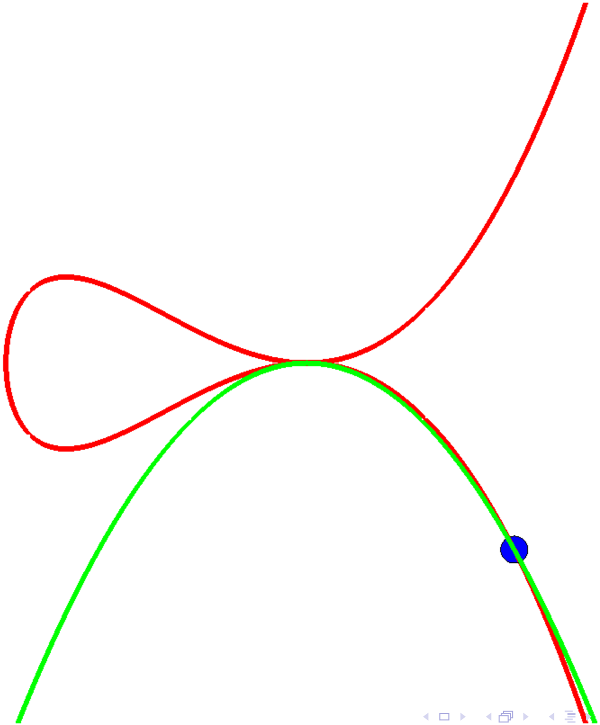

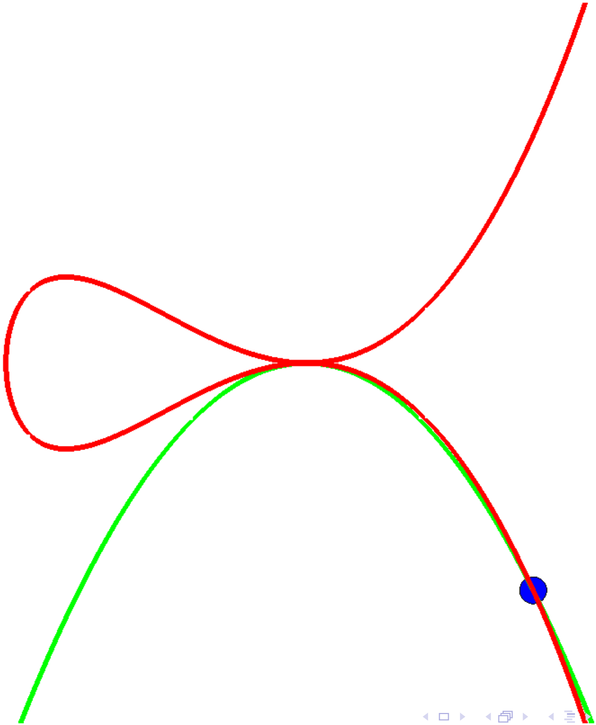

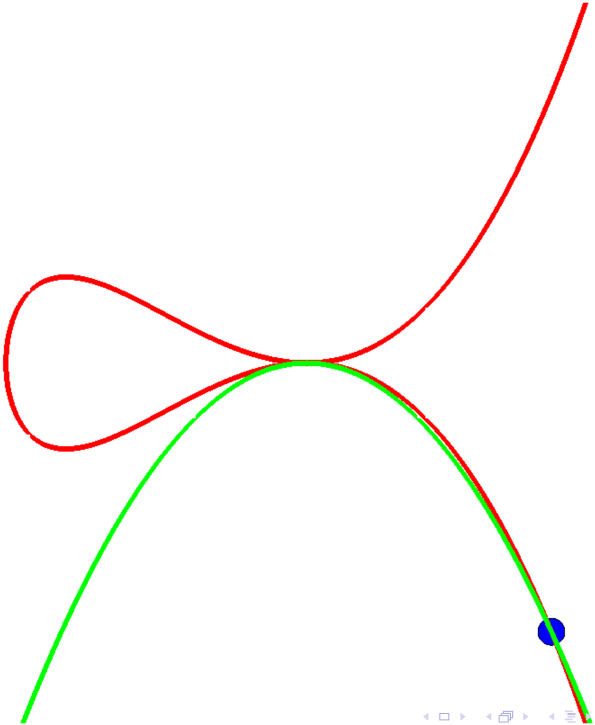

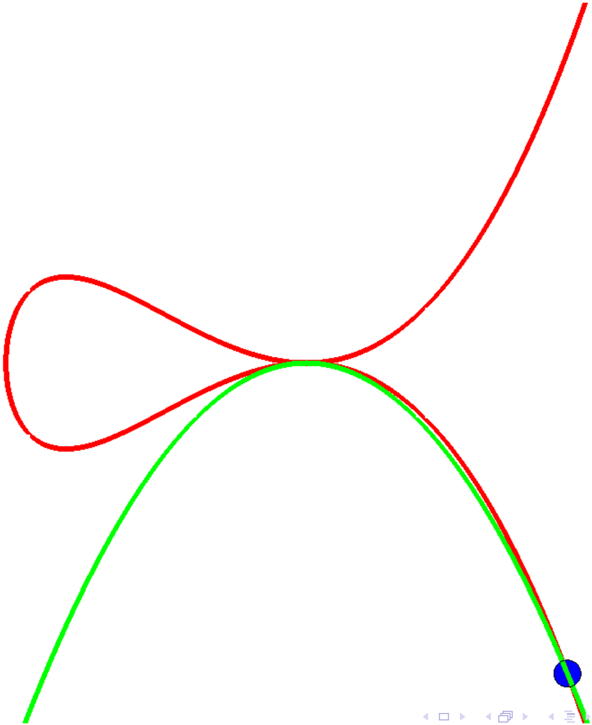

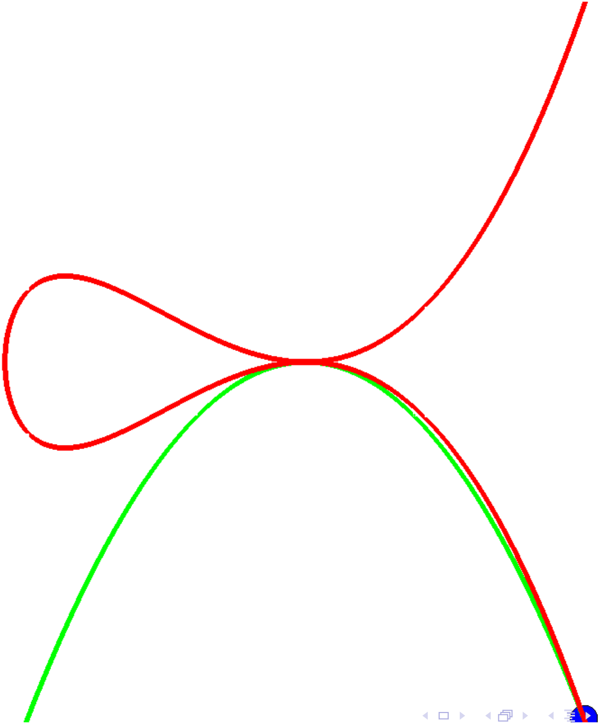

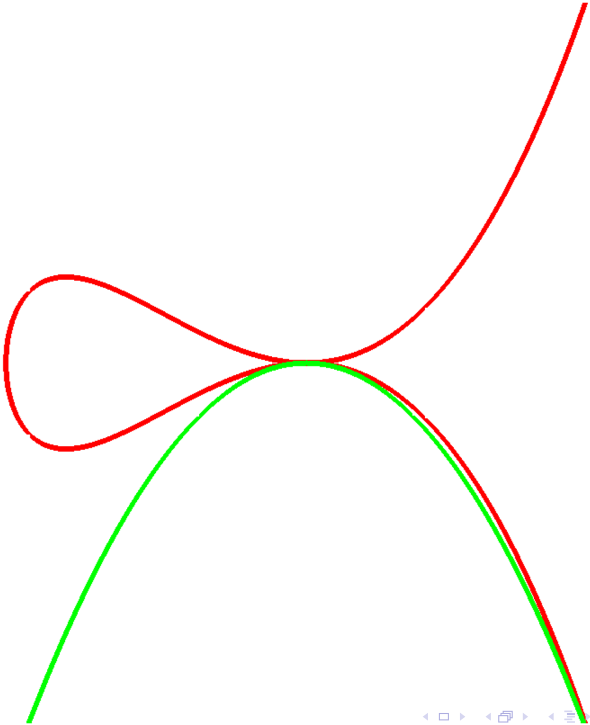

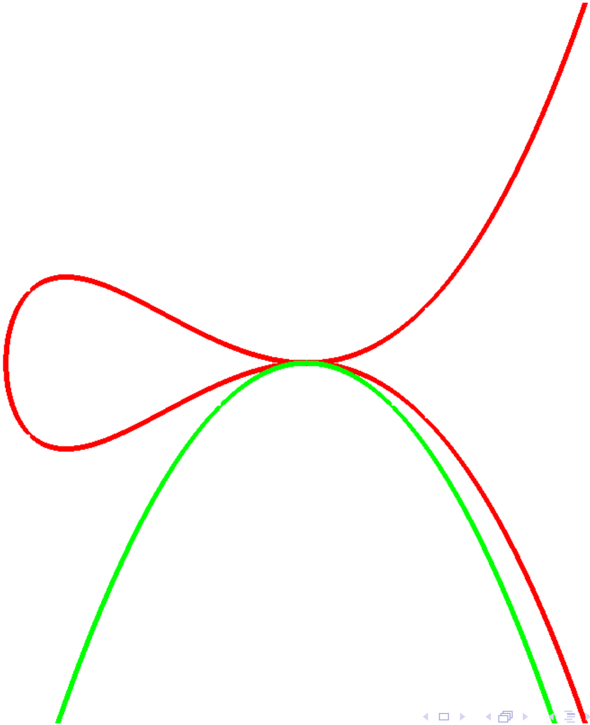

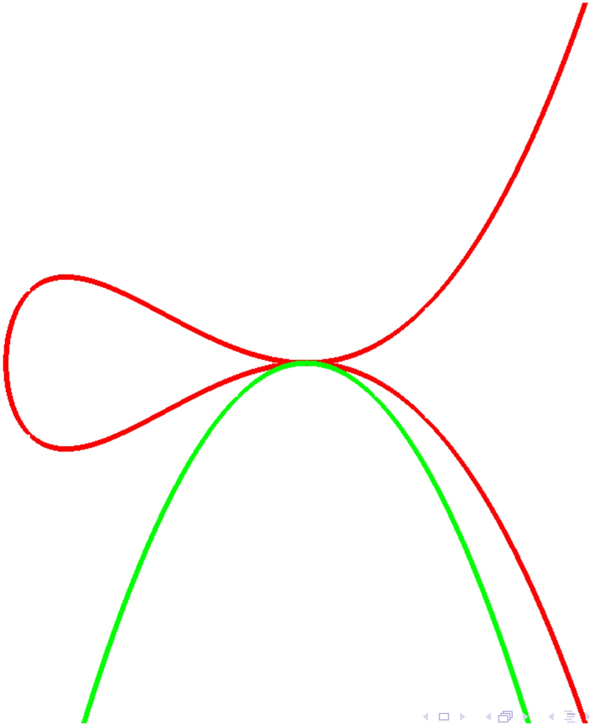

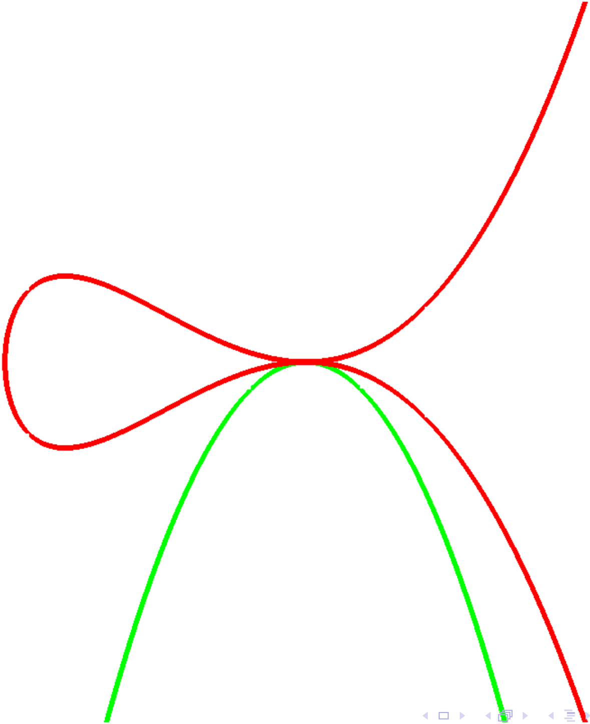

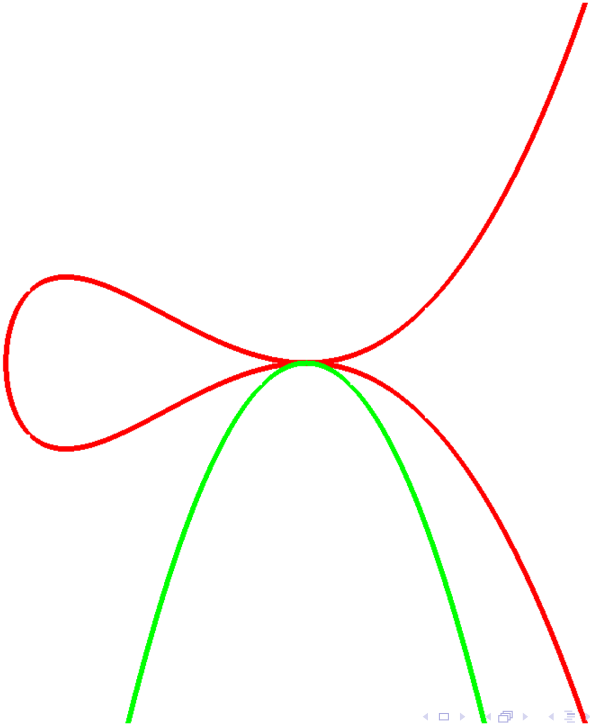

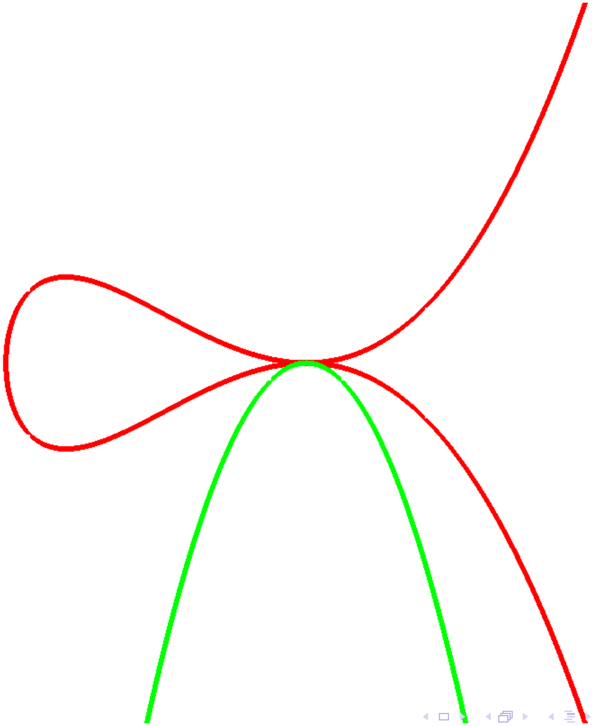

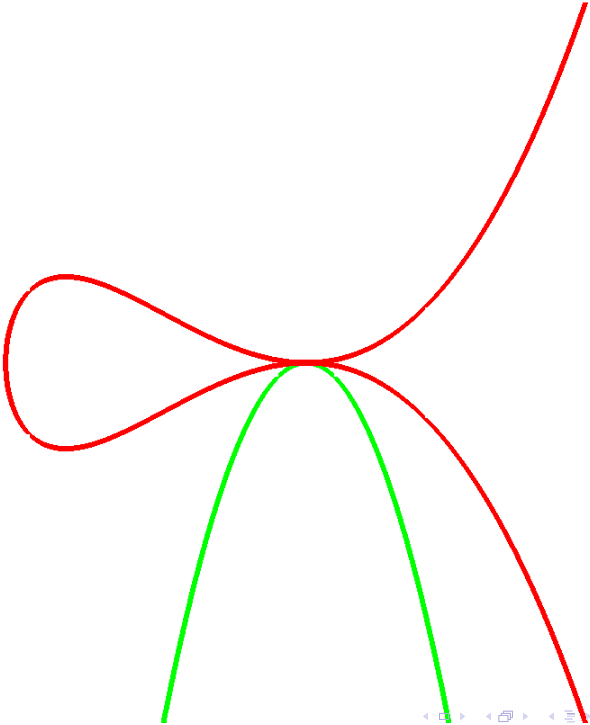

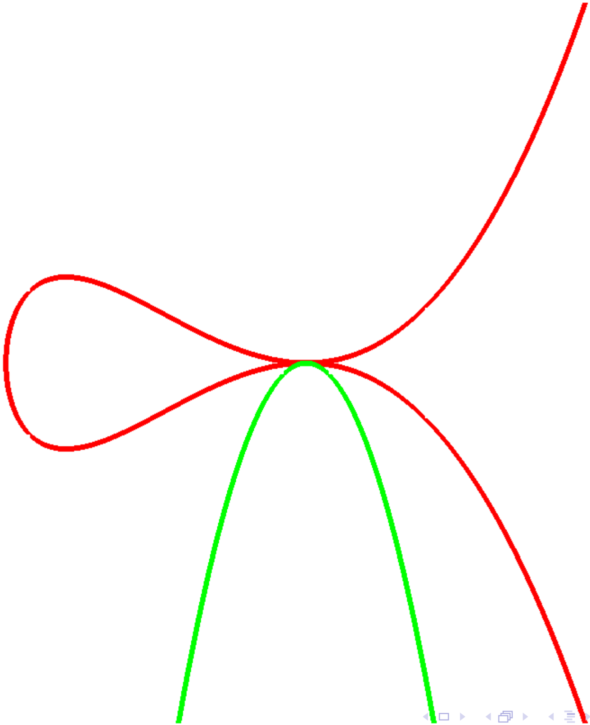

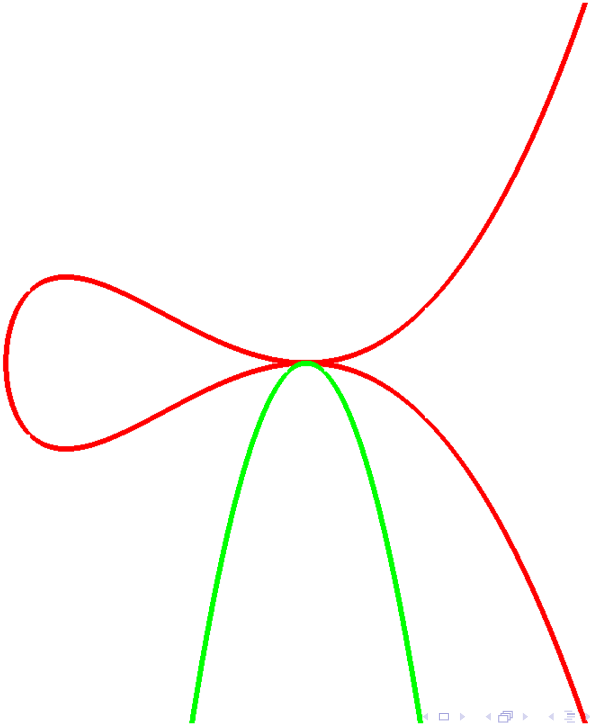

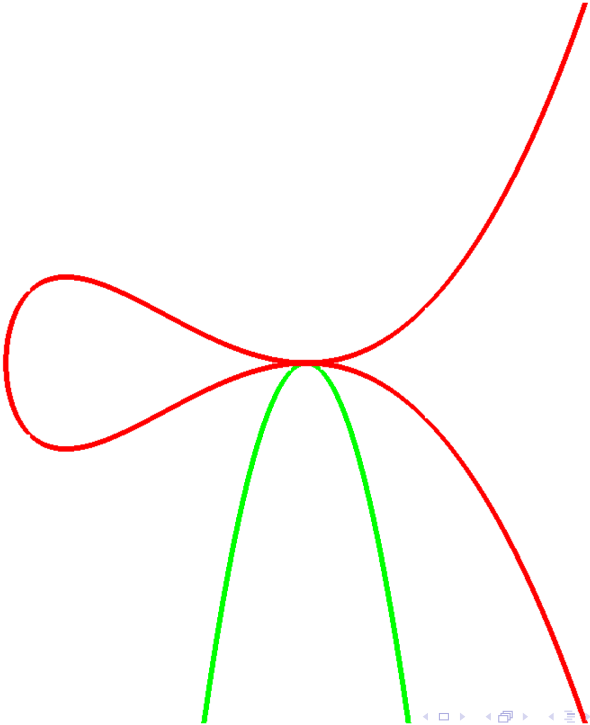

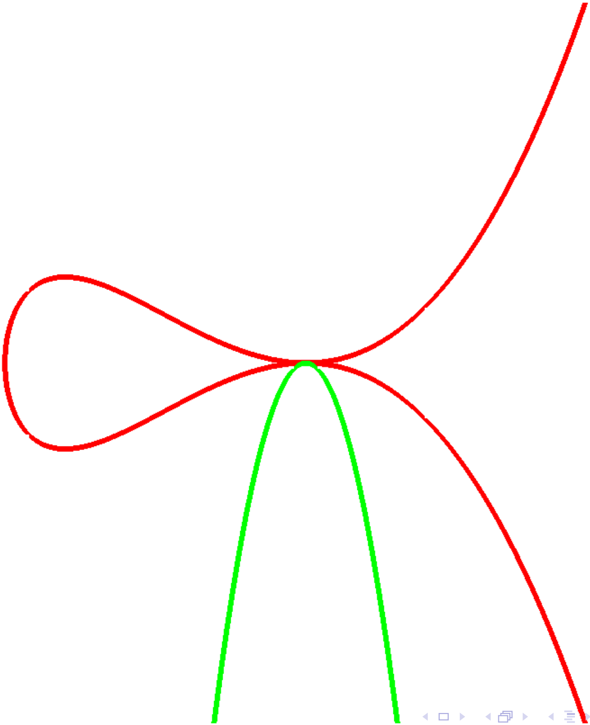

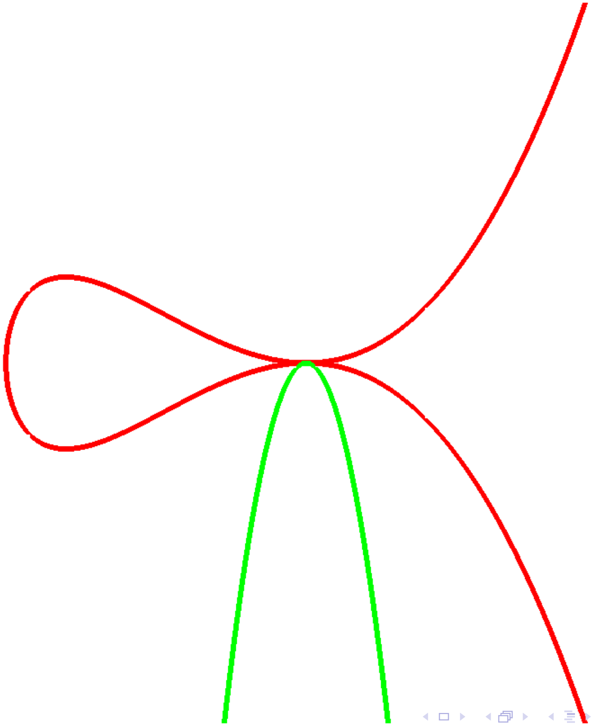

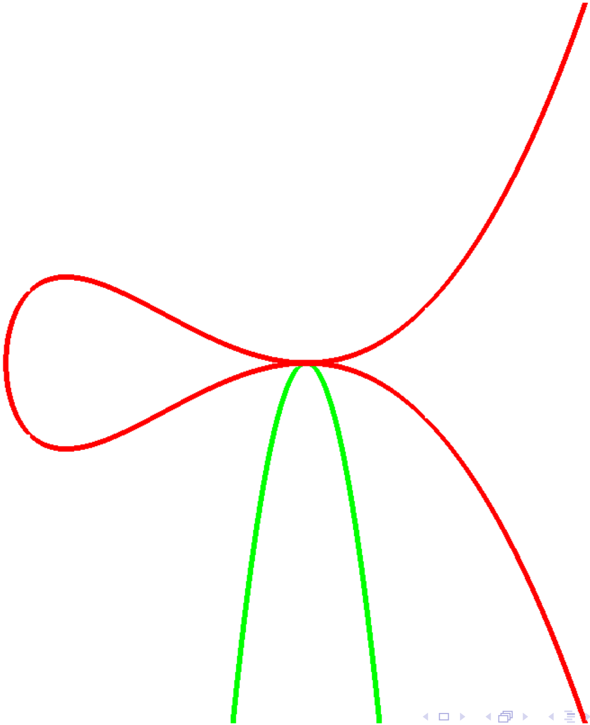

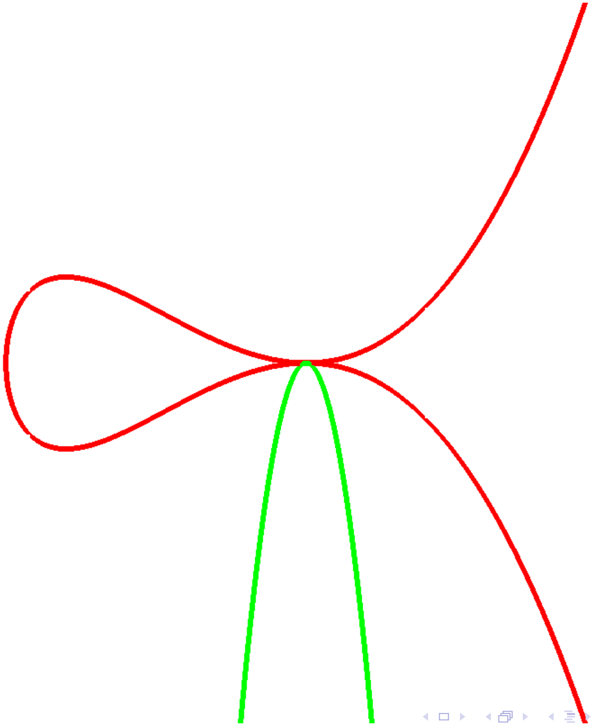

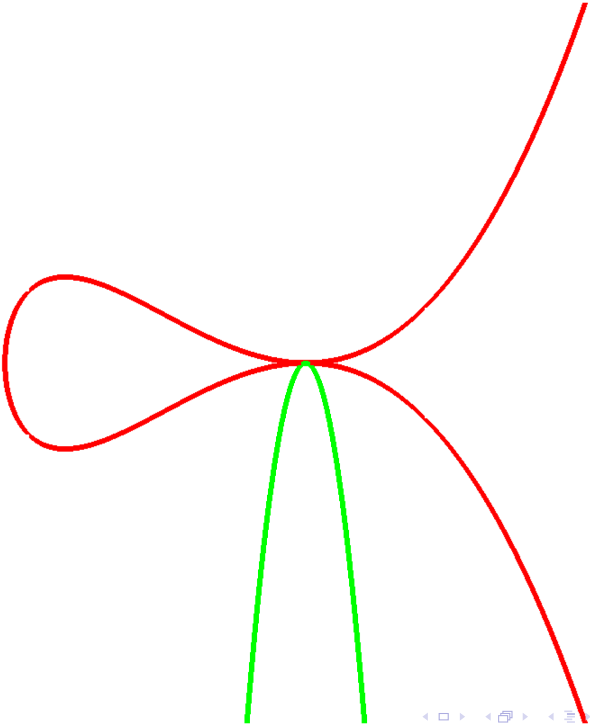

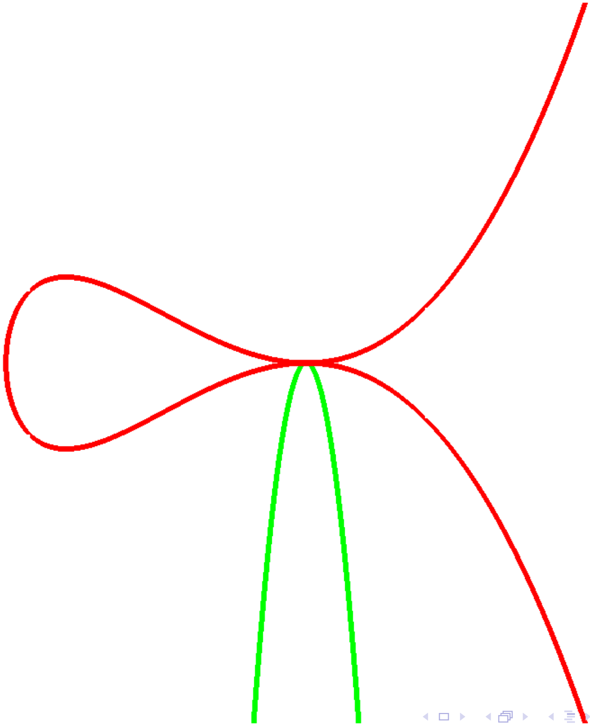

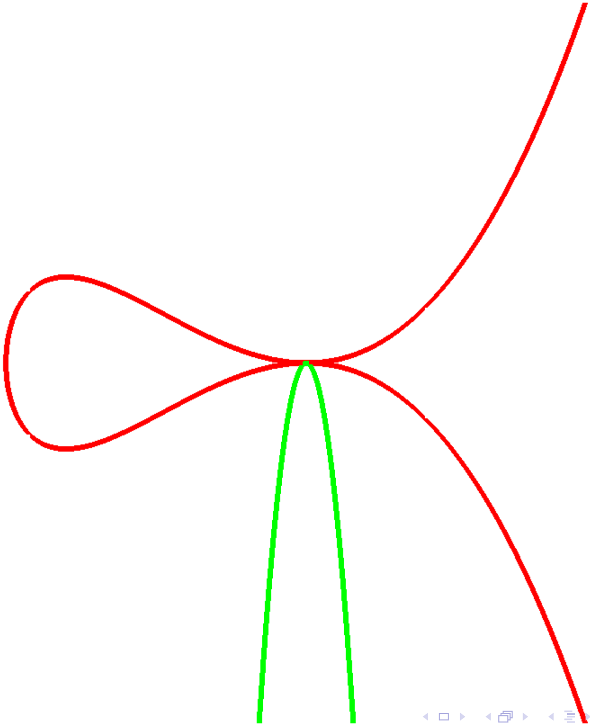

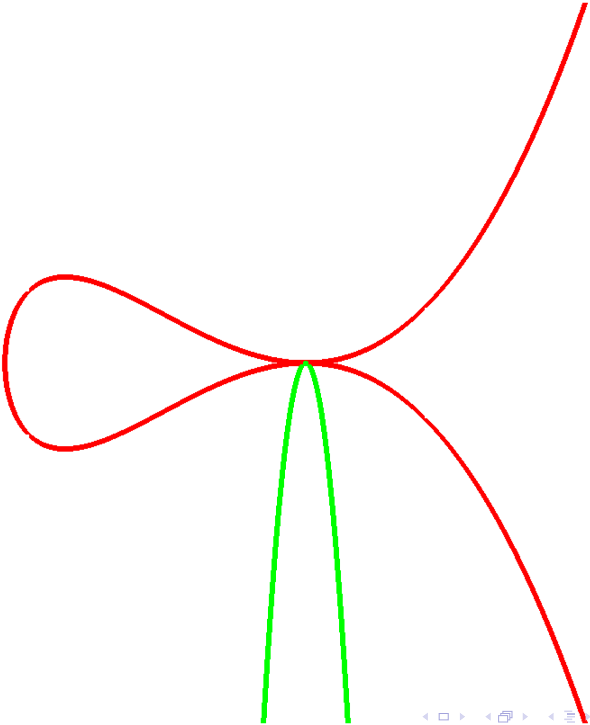

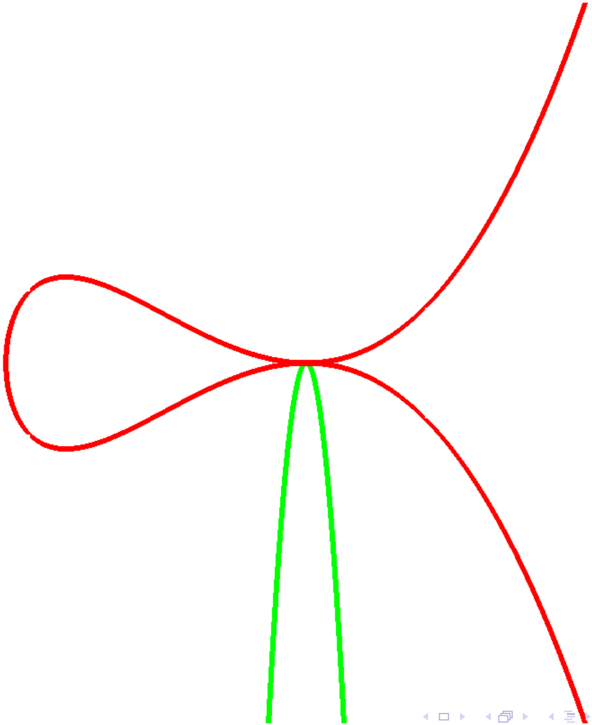

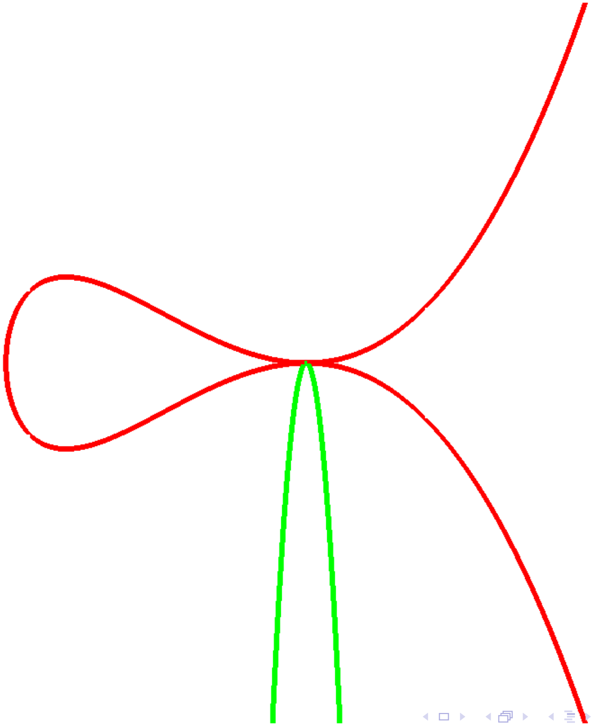

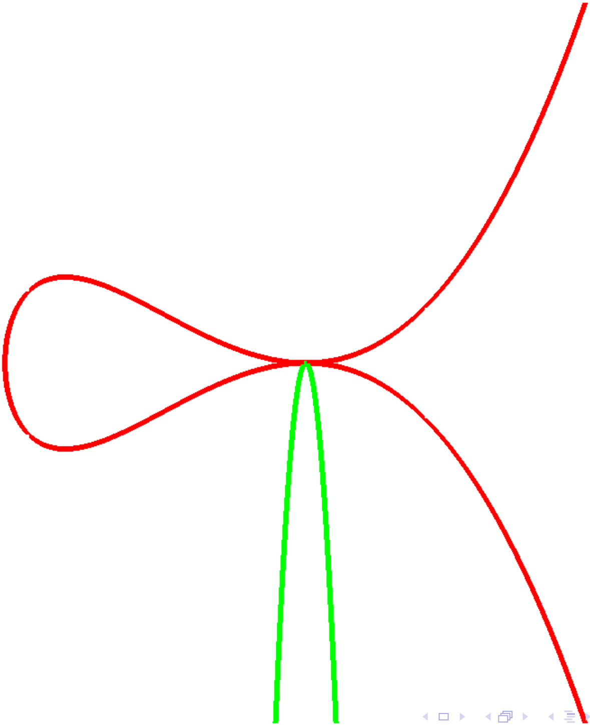

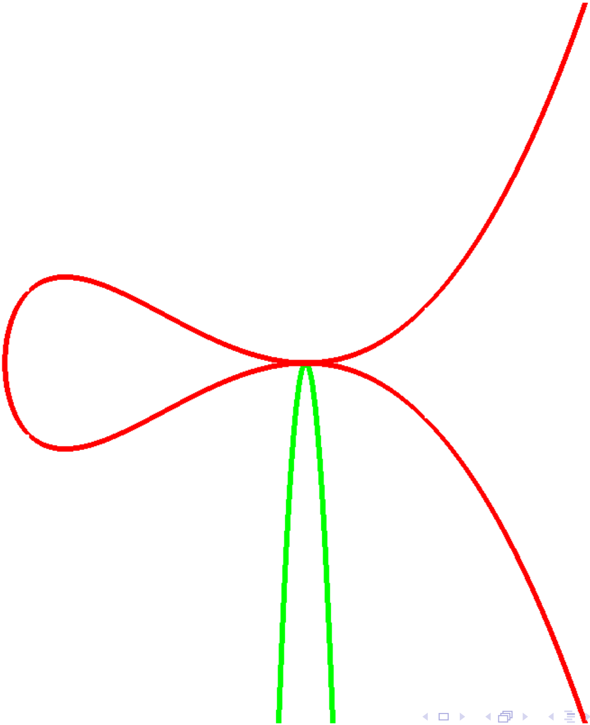

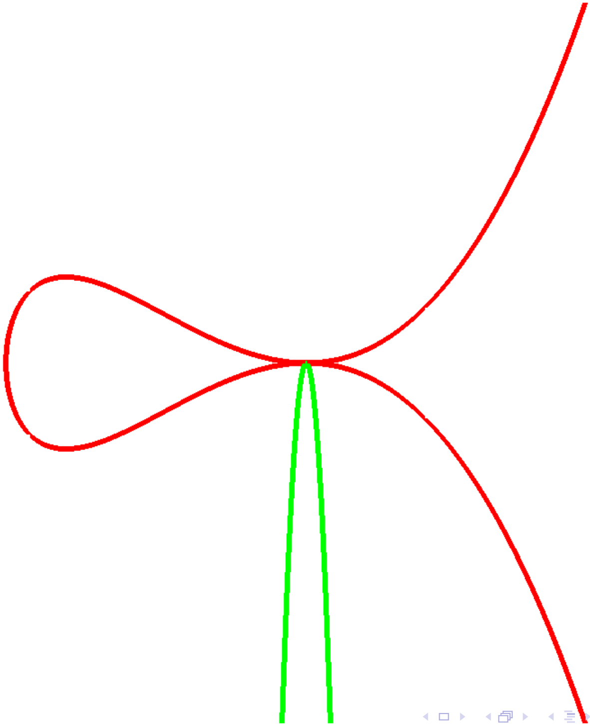

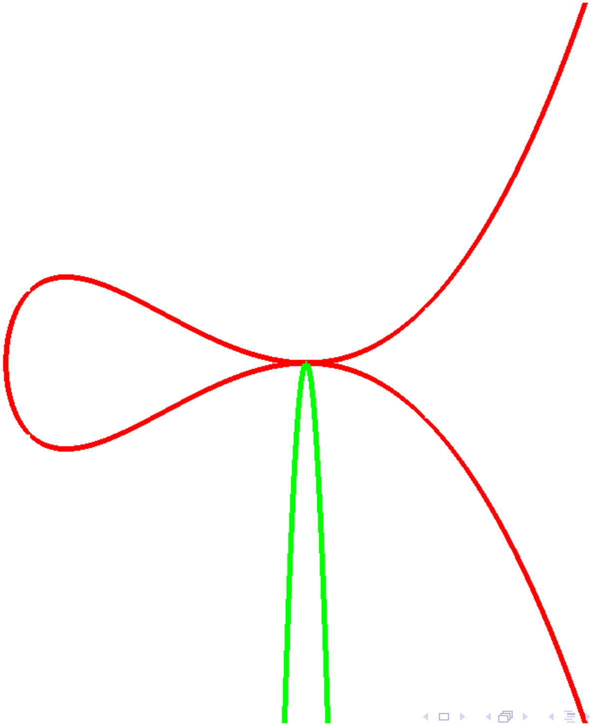

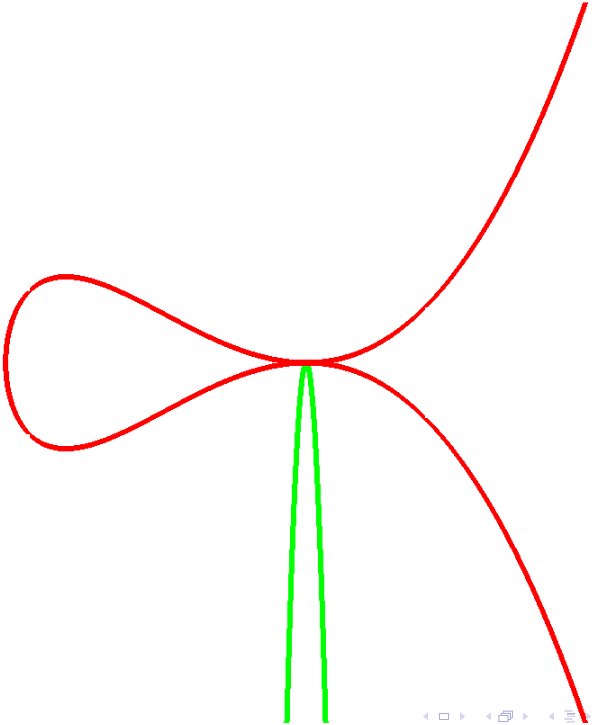

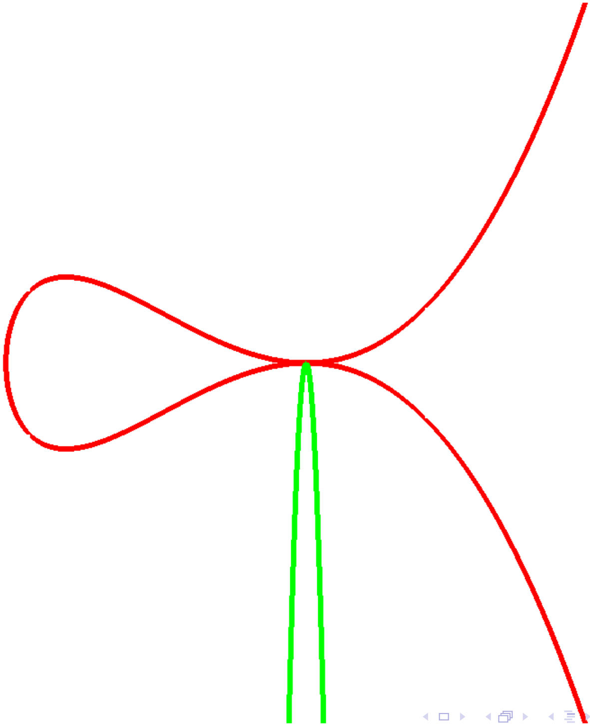

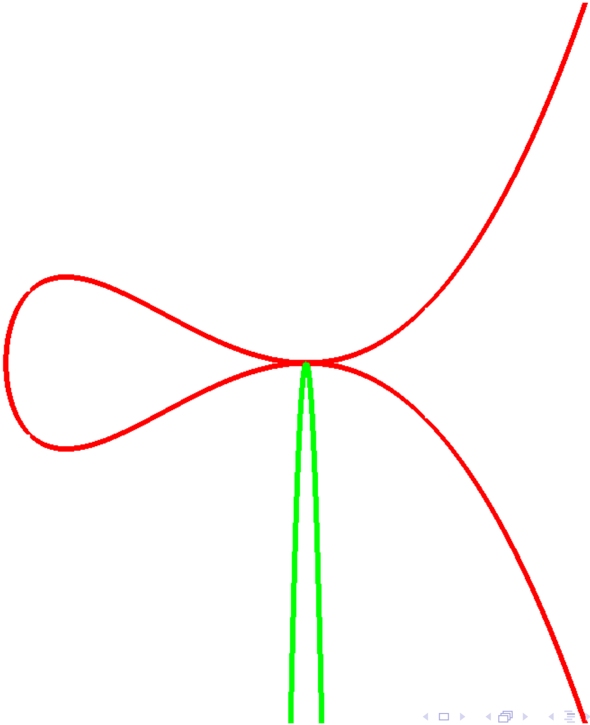

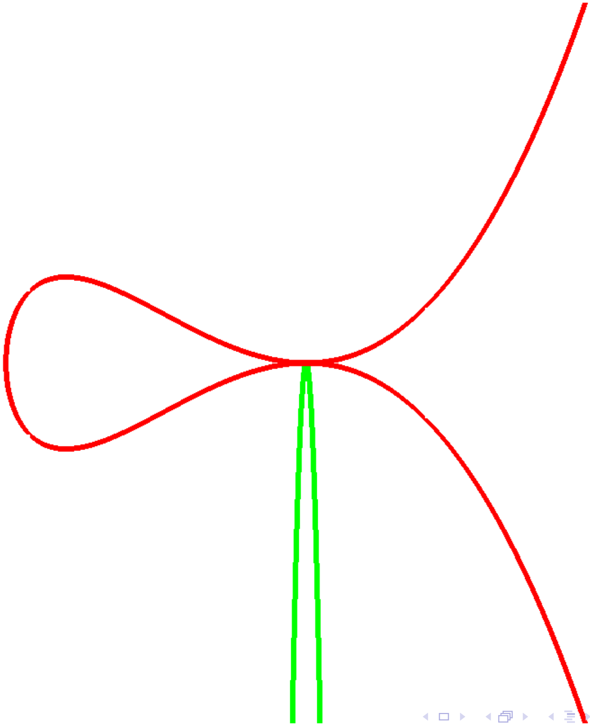

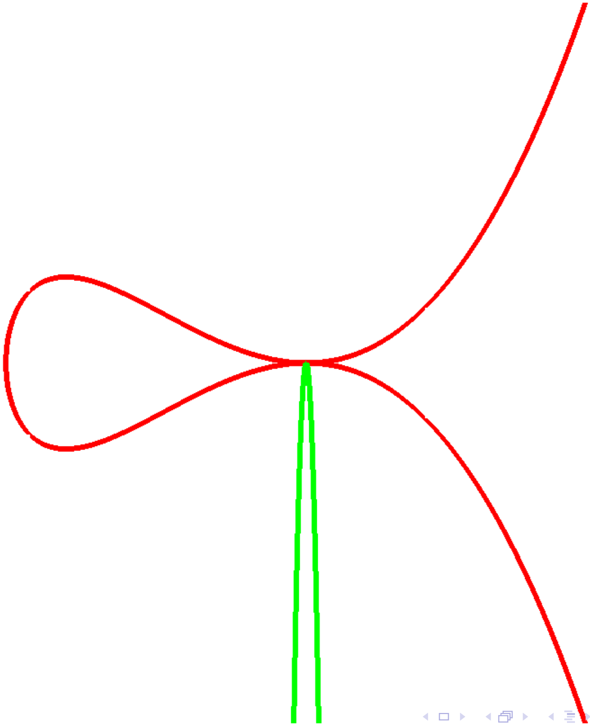

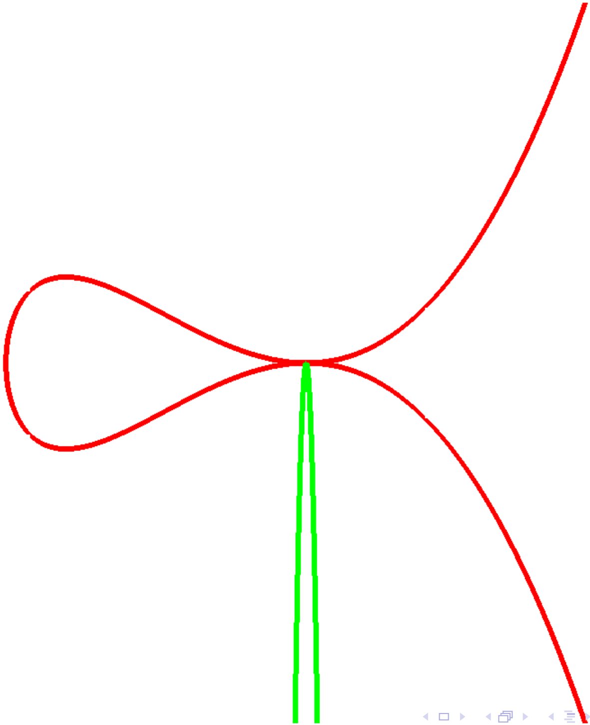

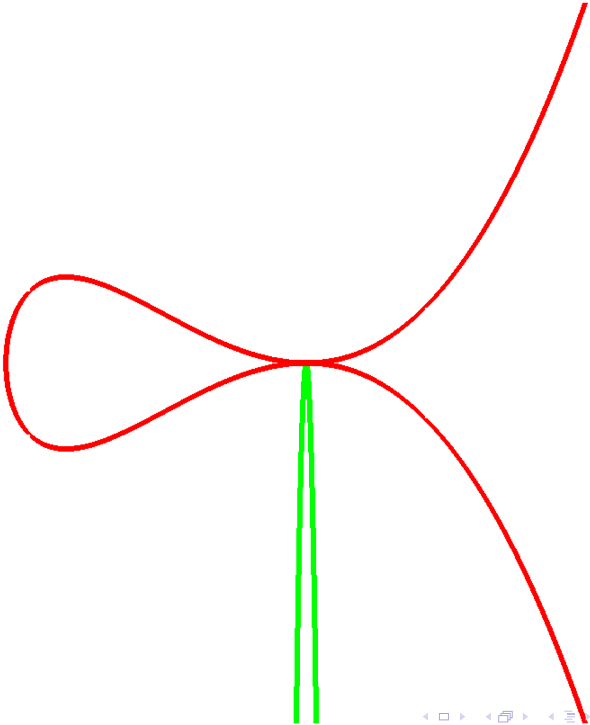

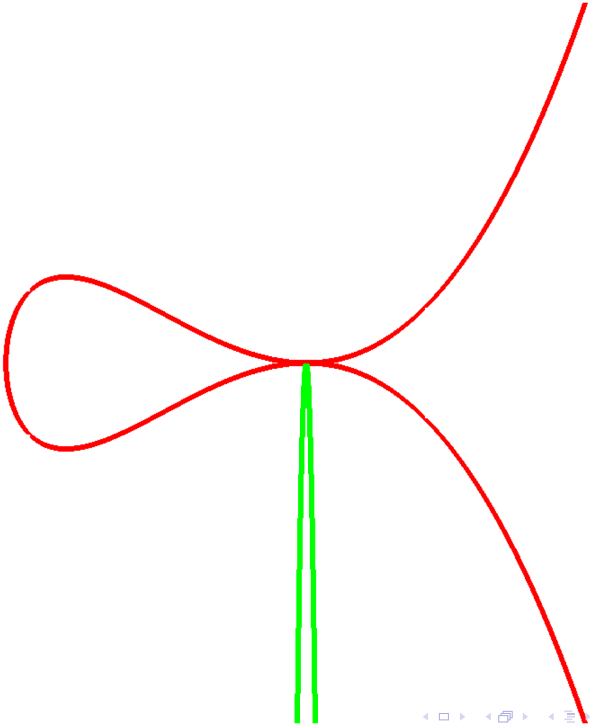

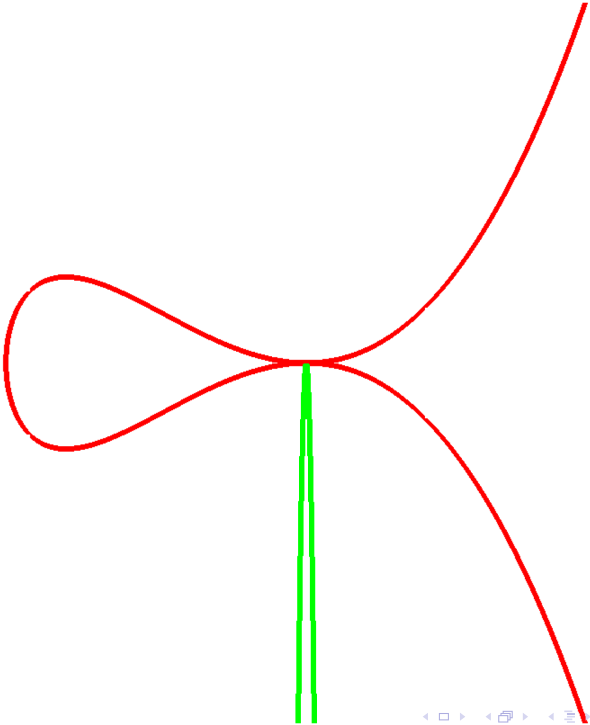

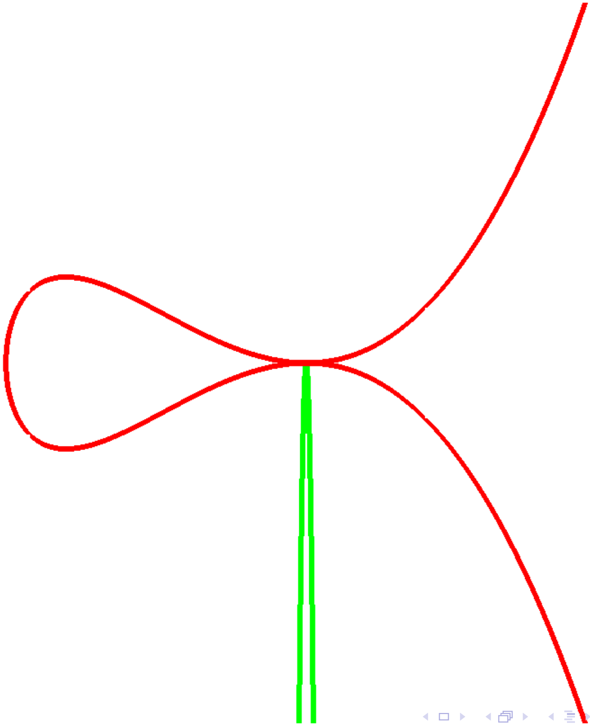

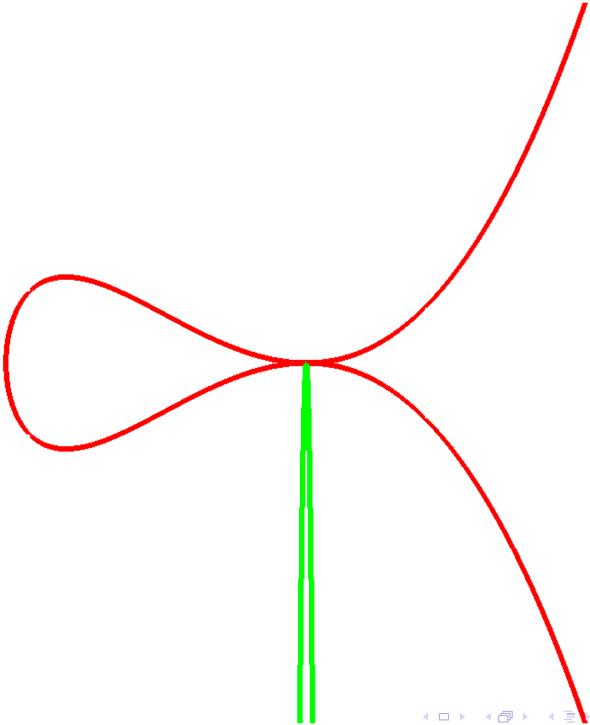

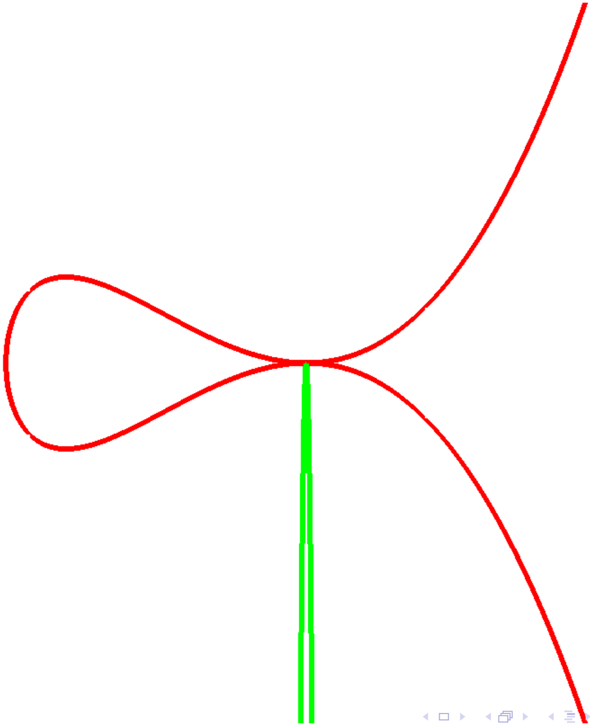

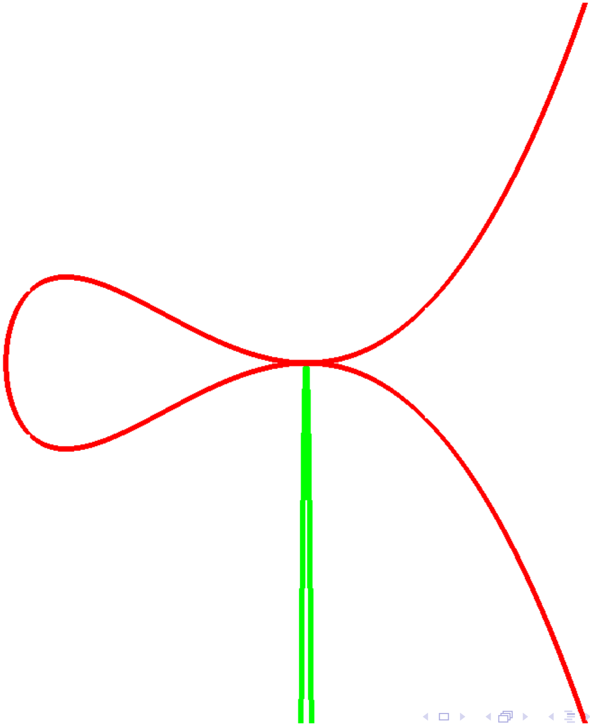

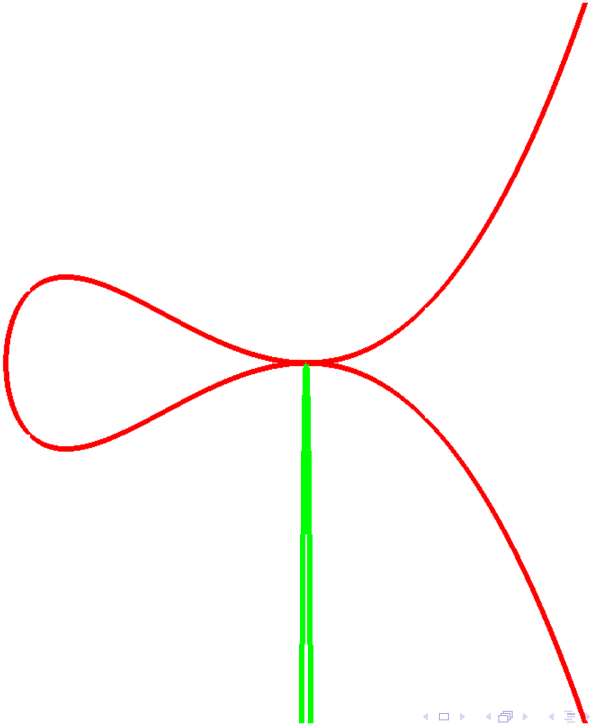

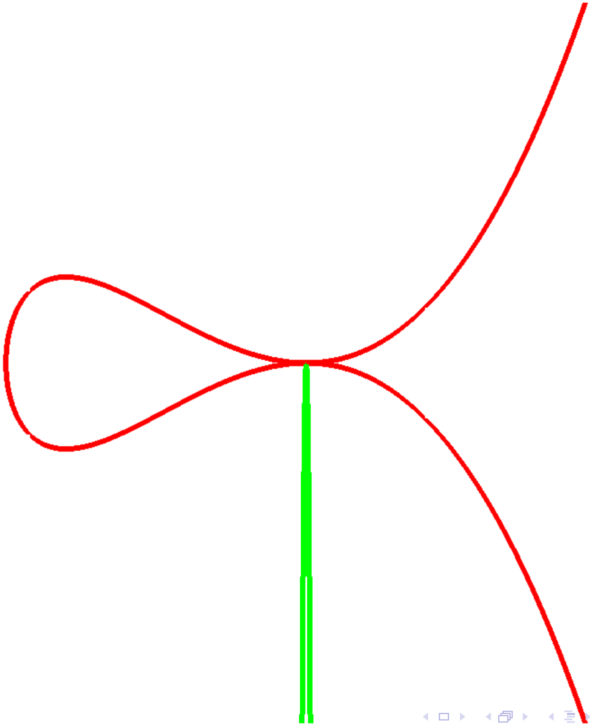

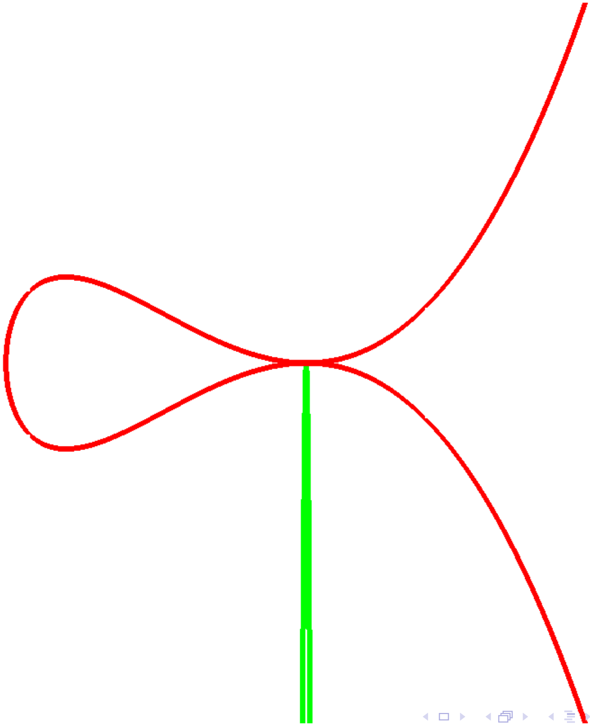

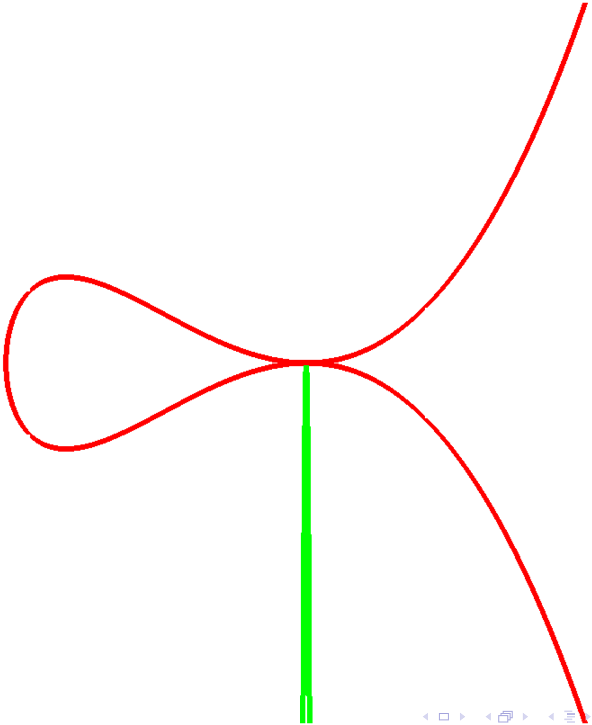

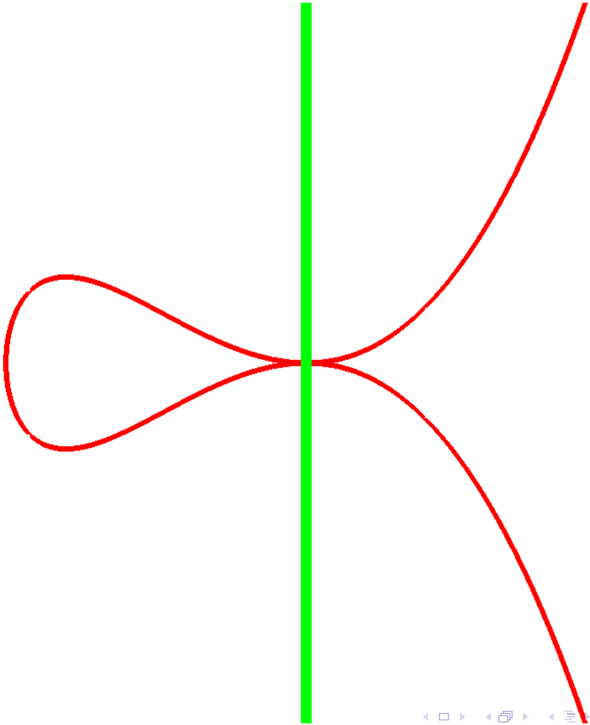

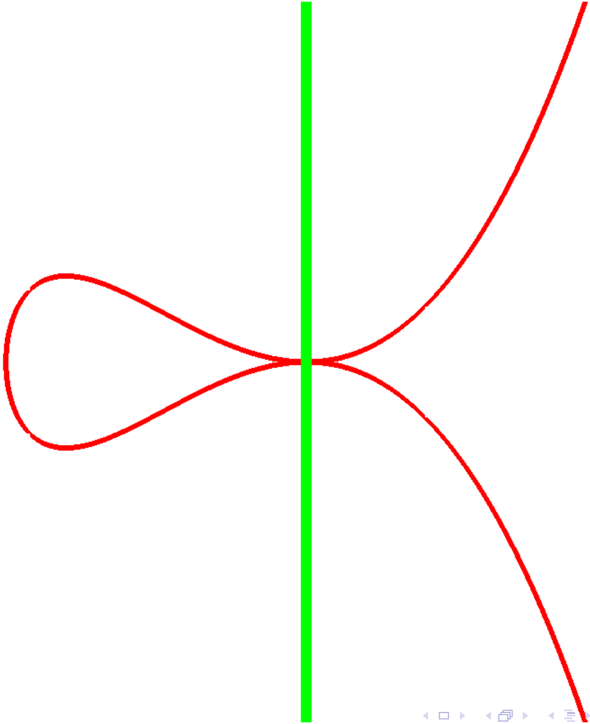

Local Techniques for Normalization

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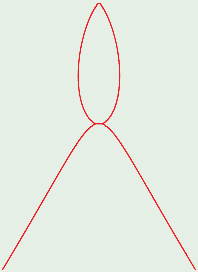
$$\overline{A} = B_1 + \ldots + B_r$$

Local Techniques for Normalization

Example

For $I = \langle x^4 + y^2(y-1)^3 \rangle$



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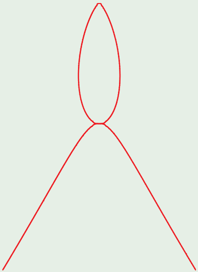

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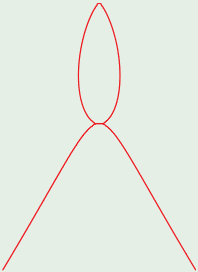

we have

$$J = \langle x, y \rangle \cap \langle x, y-1 \rangle$$

and $\overline{A} = B_1 + B_2$ with

$$B_1 = \left\langle 1,\ \frac{x^2}{y}, \frac{x^4}{y^2} \right\rangle \qquad B_2 = \left\langle 1,\ \frac{x^2}{y-1},\ \frac{x^3}{(y-1)^2} \right\rangle$$

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