Decomposition of Semigroup Algebras

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Semigroup rings

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$$B = \langle (2,0,3), (4,0,1), (0,2,3), (1,3,1), (1,2,2) \rangle \subset \mathbb{N}^3 \quad K \text{ field}$$

$$\begin{aligned} K[B] &= K[t_1^2 t_3^3, t_1^4 t_3, t_2^2 t_3^3, t_1 t_2^3 t_3, t_1 t_2^2 t_3^2] \\ &\cong K[x_0, x_1, x_2, x_3, x_4] / \langle x_1 x_2^2 - x_0 x_4^2,\ x_0 x_2 x_3^2 - x_4^4,\ x_0^2 x_3^2 - x_1 x_2 x_4^2 \rangle \end{aligned}$$

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- Decompose $K[B]$ into simpler pieces.
- Fast algorithm to compute the regularity of $K[B]$ via decomposition.
- Determine ring theoretic properties of $K[B]$ via decomposition.
- Verify conjectured bounds on the regularity.

Decomposition theorem

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Proof.

$$R[B] = \bigoplus_{g \in G} I'_g \quad \text{with} \quad I'_g = R \cdot \left\{ t^b \mid b \in B \cap g \right\} \subset R[B]$$

If we choose for each $g \in G$ an $h_g \in g$ then as $R[A]$-modules

$$I'_g \cong I_g := R \cdot \left\{ t^{b-h_g} \mid b \in B \cap g \right\} \underset{b-h_g \in G(A)}{\subset} R[G(A)]$$

□

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$$B_A = \{x \in B \mid x - a \notin B \ \forall a \in A \setminus \{0\}\}$$

is the unique minimal subset of B with

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$$K[B] = K[A] \cdot \left\{ t^b \mid b \in B_A \right\}.$$

For computations we need that B_A is finite.
Write $C(B)$ for the positive rational cone generated by B.

Lemma

$K[B]$ is finitely generated $K[A]$-module iff $C(A) = C(B)$.

Note: If B_A is finite then also $G = G(B)/G(A)$.

Algorithm for minimal generators

For simplicity $A = \langle e_1, ..., e_d \rangle \subset B = \langle b_1 = e_1, ..., b_d = e_d, b_{d+1}, ..., b_n \rangle$.

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Return: $B_A = \{\deg(v_i) \mid i\}$.

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$$K[x_0, ..., x_4] / \langle x_0, ..., x_3, I_B \rangle = K[x_0, ..., x_4] / \langle x_0, ..., x_3, x_4^4 \rangle =_K \langle 1, \bar{x}_4, \bar{x}_4^2, \bar{x}_4^3 \rangle$$

$$B_A = \{(0,0,0), (1,2,2), (2,4,4), (3,6,6)\}$$

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$$R[B] \cong \bigoplus_{g \in G} I_g(-h_g)$$

with monomial ideals $I_g \subset K[A]$ *and twists* $h_g \in G(B)$.

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Return: $h_g = g' + \sum_{j=1}^{d} c'_{g,j} \cdot e_j \qquad I_g = K[A] \cdot \left\{ t^{v - h_g} \mid v \in \Gamma_g \right\}$

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Note: $v - h_g = \sum_{j=1}^{d} \underbrace{(c_{v,j} - c'_{g,j})}_{\geq 0} \cdot e_j \in A$, hence I_g is an ideal in $K[A]$.

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Row reduction gives $G(B)/G(A) = \{\overline{(0,0,0)}, \overline{(5,0,0)}\}$, hence

$$B_A = \{(0,0,0), (2,4,4)\} \cup \{(1,2,2), (3,6,6)\}$$

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	$(2,0,3)$	$(4,0,1)$	$(0,2,3)$	$(1,3,1)$
$(0,0,0)-(0,0,0)=$	0	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$(2,4,4)-(0,0,0)=$	$\mathbf{-1}$	1	2	$\mathbf{0}$
$(1,2,2)-(5,0,0)=$	0	$\mathbf{-1}$	$\mathbf{1}$	$\mathbf{0}$
$(3,6,6)-(5,0,0)=$	$\mathbf{-1}$	0	3	$\mathbf{0}$

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$(1,2,2)-(5,0,0)=$	0	$\mathbf{-1}$	$\mathbf{1}$	$\mathbf{0}$
$(3,6,6)-(5,0,0)=$	$\mathbf{-1}$	0	3	$\mathbf{0}$

Twists $h_{(0,0,0)} = (0,0,0) - (2,0,3) = (-2,0,-3)$

$$h_{(5,0,0)} = (5,0,0) - (2,0,3) - (4,0,1) + (0,2,3) = (-1,2,-1)$$

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$$B_A = \begin{matrix} \{(0,0,0), (2,4,4)\} & \cup & \{(1,2,2), (3,6,6)\} \\ h_{(0,0,0)} = (-2,0,-3) & & h_{(5,0,0)} = (-1,2,-1) \end{matrix}$$

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So the corresponding ideals are generated by

$$\begin{array}{ll} (0,0,0)+(2,0,3)=(2,0,3) & (1,-2,1)+(1,2,2)=(2,0,3) \\ (2,4,4)+(2,0,3)=(4,4,7) & (1,-2,1)+(3,6,6)=(4,4,7) \end{array}$$

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Hence, by $t_1^2 t_3^3 = x_0$ and $t_1^4 t_2^4 t_3^7 = x_1 x_2^2$ we get

$$K[B] = \left\langle x_0, x_1 x_2^2 \right\rangle (2,0,3) \oplus \left\langle x_0, x_1 x_2^2 \right\rangle (1,-2,1)$$

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Via the decomposition of semigroup algebras we can develop a very fast algorithm to determine the regularity of $K[B]$.

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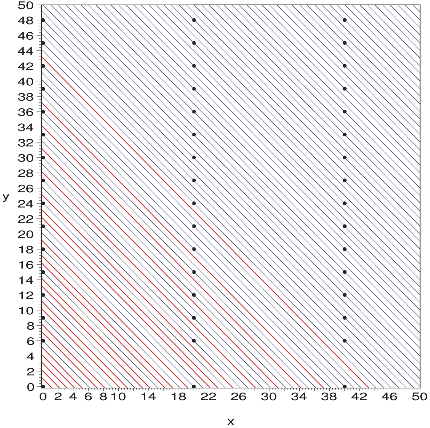
$$F(6,9,20) = 43 \leq (\operatorname{reg} K[B] - 1)\, a_3 - 1 = (7-1)\cdot 20 - 1 = 119$$



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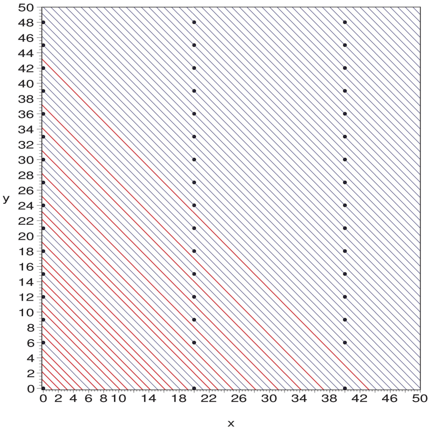

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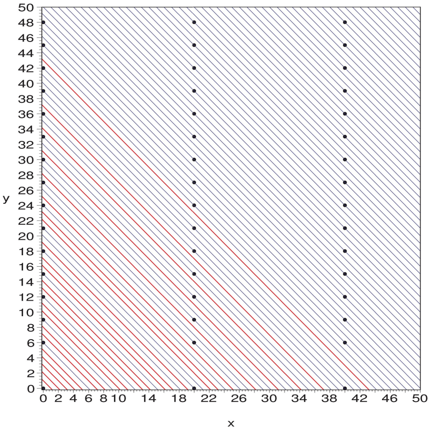

More serious applications:

- $\operatorname{Syz}_i M$ can be generated in degree $\leq \operatorname{reg} M + 1$.
- For $t \geq \operatorname{reg} M + 1$ for the Hilbert function it holds $H_M(t) = P_M(t)$ with the Hilbert polynomial $P_M \in \mathbb{Q}[t]$.

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Key observation: Minimal graded res of I_g has length at most $|\operatorname{Hilb}(A)|$.

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In the above example

$$K[B] = I(2,0,3) \oplus I(1,-2,1) \text{ with}$$
$$I = \left\langle x_0, x_1 x_2^2 \right\rangle \subset K[A] \cong K[x_0,x_1,x_2,x_3] / \left\langle x_1^2 x_2^3 - x_0^3 x_3^2 \right\rangle$$

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hence $\operatorname{reg} I = 4$. The group homomorphism is given by $\deg b = (b_1 + b_2 + b_3)/5$ and therefore

$$\operatorname{reg} K[B] = \max\{4-1, 4-0\} = 4.$$

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MA	.073	.089	.095	.10	.13	.14	.14	.19	.16
RES	.0099	.0089	.011	.013	.020	.046	.18	1.1	6.8
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c	10	11	12	13	14	15	16	17	18
MA	.21	.26	.22	.26	.29	.30	.31	.36	.47
RES	30	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
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- $K = \mathbb{F}_{101}$, $B \subset \mathbb{N}^4$ with $c + 4$ random generators, coordinate sum 5:

c	12	16	20	24	28	32	36	40	44	48	52
MA	3.8	13	.69	2.2	1.7	1.9	1.5	4.4	6.0	8.9	13
BG	46	150	380	840	940	$*$	$*$	$*$	$*$	$*$	$*$

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4. *$K[B]$ is normal $\iff \forall x \in B_A\ \exists \lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ with $0 \leq \lambda_i < 1$ s.t. $x = \sum_{i=1}^{d} \lambda_i e_i$.*

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4. $K[B]$ *is normal* $\Longleftrightarrow \forall x \in B_A\ \exists \lambda_1, \ldots, \lambda_d \in \mathbb{Q}$ *with* $0 \leq \lambda_i < 1$ *s.t.* $x = \sum_{i=1}^d \lambda_i e_i$.
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Conjecture (Eisenbud-Goto, 1984)

If K is algebraically closed and $I \subset R = K[x_1, ..., x_n]$ is a homogeneous prime ideal then for $S = R/I$

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    - are seminormal (Nitsche)

Regularity bounds

Proposition

The Eisenbud-Goto conjecture holds provided that the minimal generators of B in $\mathbb{N}^d$ have fixed coordinate sum α for $d = 3$ and $\alpha \leq 5$, for $d = 4$ and $\alpha \leq 3$, as well as for $d = 5$ and $\alpha = 2$.

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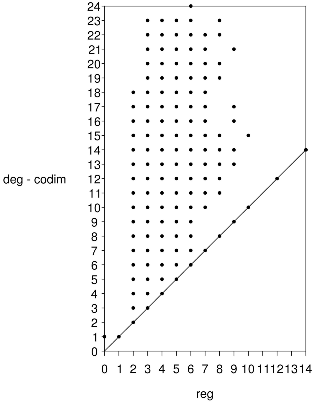

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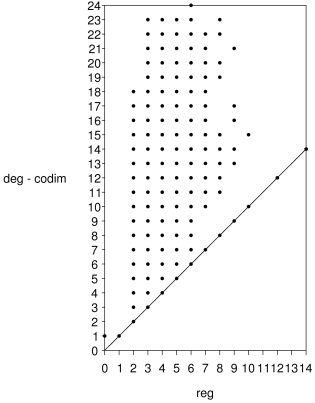

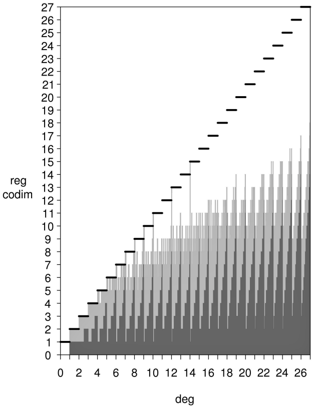

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