Tropical mirror symmetry

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Calabi-Yau varieties and mirror symmetry

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Deformations of

complex structure	symplectic structure

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Tangent spaces

$H^1(T_X) = H^{2,1}(X)$	$\cong$	$H^{1,1}(X^\vee)$
by Bogomolov-Tian-Todorov if		Moser

$H^0(T_X) = H^{2,0}(X) = H^{1,0}(X) = 0$

Calabi-Yau varieties and mirror symmetry

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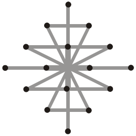

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
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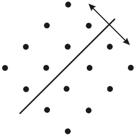

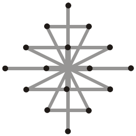

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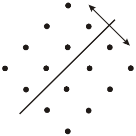
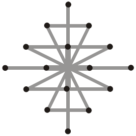


$$H^{2,1}(X) \quad \cong \quad H^{1,1}(X^\vee)$$

mirror map induces equality

$$\mathbb{B}_g^X(q) \quad = \quad \mathbb{A}_g^{X^\vee}(Q)$$

(differential equations) (# of genus g curves)



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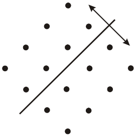

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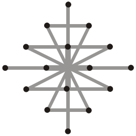

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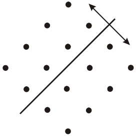

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interpreting lattice points as

Deformations Divisor classes

$\left(H^1\left(X^\vee, \mathcal{O}^*_{X^\vee}\right) = H^2\left(X^\vee, \mathbb{Z}\right)\right)$

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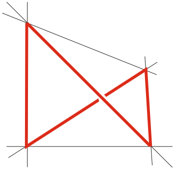

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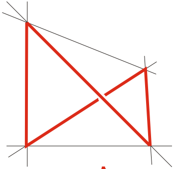

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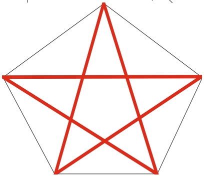

$$X_0 = \{x_0x_1 = x_1x_2 = x_2x_3 = x_3x_4 = x_4x_0 = 0\} \subset \mathbb{P}^4$$

X_t by structure theorem of Buchsbaum-Eisenbud

Structure results for Gorenstein rings

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Definition

$X \subset \mathbb{P}^n$, $\operatorname{codim} X = 3$ is called Pfaffian subscheme if X is the degeneracy locus of a skew symmetric map $\varphi : \mathcal{E}(-t) \longrightarrow \mathcal{E}^*$, where $\mathcal{E}$ is rank $2k+1$ vector bundle on $\mathbb{P}^n$, φ is generically of rank $2k$, degenerates in expected codim 3 to rank $2k-2$.

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Theorem (Buchsbaum-Eisenbud)

Has loc. free resolution ($\psi = \operatorname{Pfaff}_{2k} \varphi$, $s = c_1(\mathcal{E}) + kt$)

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-t-2s) \to \mathcal{E}(-t-s) \xrightarrow{\varphi} \mathcal{E}^*(-s) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

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Theorem (Walter)

Gorenstein, codim 3, $\omega_X^\circ \cong \mathcal{O}_X(l)$, *tech. cond.* $\Longrightarrow$ *Pfaffian*

Structure results for Gorenstein rings

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$$0 \longrightarrow R^1(-5) \longrightarrow R^5(-3) \longrightarrow R^5(-2) \longrightarrow R/I$$

$$\begin{pmatrix} -x_0x_4 & x_3x_4 & -x_2x_3 & x_1x_2 & -x_0x_1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & x_1 & x_3 & 0 \\ 0 & 0 & 0 & x_0 & x_2 \\ -x_1 & 0 & 0 & 0 & x_4 \\ -x_3 & -x_0 & 0 & 0 & 0 \\ 0 & -x_2 & -x_4 & 0 & 0 \end{pmatrix} = 0$$

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Theorem (Kustin-Miller unprojection)

R/I Gorenstein codim c, $I \subset J$, $\phi : J/I \to R/I \Longrightarrow$

$$R[T] \quad / \quad \langle I,\ T \cdot u - \phi(u) \mid u \in J \rangle \text{ is Gorenstein codim } c+1$$

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$Y = \mathrm{TV}(\Sigma)$ a $\mathbb{Q}$-Gorenstein toric Fano variety, where $\Sigma = \mathrm{Fan}\,(\Delta^*)$ over a Fano polytope in $N_\mathbb{R} = N \otimes \mathbb{R}$, $N = \mathbb{Z}^n$, $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$.

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Theorem (Cox)

Subschemes of $Y \rightleftarrows$ saturated (all ass. primes exist in Y) ideals of S.

Depending on Y, replace S by Picard-Cox ring $R = \bigoplus_{\alpha \in \mathrm{Pic}(Y)} S_\alpha$.

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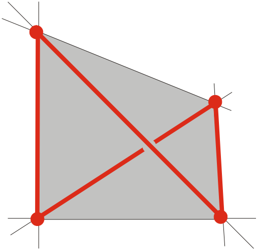

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Example

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(Loading torus.gif)

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Space of homogeneous weight vectors on S:

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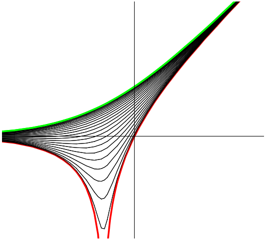
For degeneration $\mathfrak{X}$ given by I with special fiber given by I_0: special fiber Gröbner cone

$$C_{I_0}\,(I) = \mathrm{cl}\left\{w \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \mathrm{in}_w\,(I) = I_0\right\}$$

Amoebas and tropical varieties

Amoeba of $V \subset (\mathbb{C}^*)^n$ is its image under

$$\begin{array}{lccc} \log_t : & (\mathbb{C}^*)^n & \longrightarrow & \mathbb{R}^n \\ & (z_i) & \longmapsto & (-\log_t |z_i|) \end{array}$$



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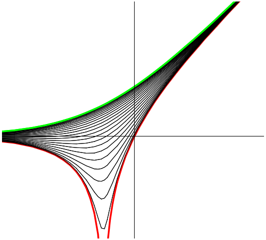

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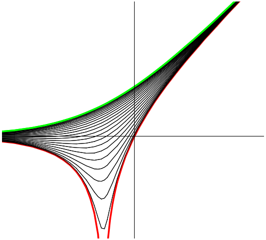

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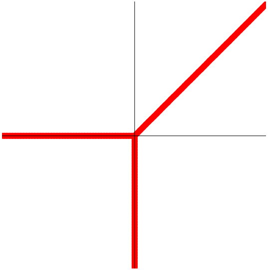

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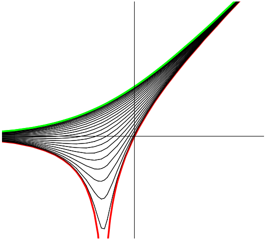

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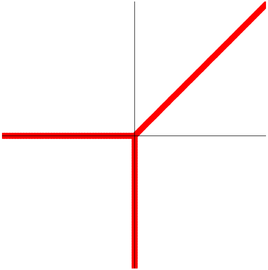

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$$\operatorname{val}(V_K(I)) = \{w \in \mathbb{R}^n \mid in_w(I) \text{ contains no monomial}\}$$

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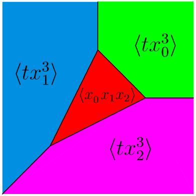

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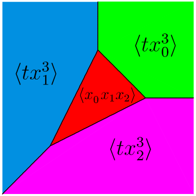

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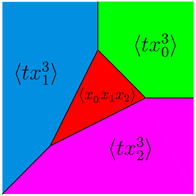

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$$\operatorname{trop}(I) = BF(I) \cap \{w_t = 1\}$$

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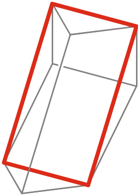

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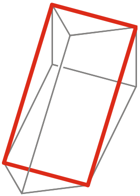

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Lemma

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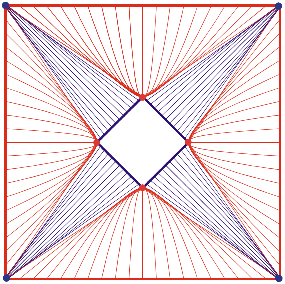

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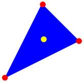
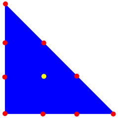
Duality of reflexive polytopes  $\longleftrightarrow$ 

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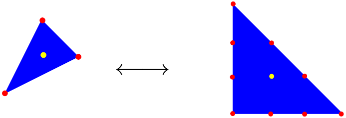
Duality of reflexive polytopes 

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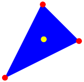
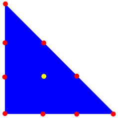
Duality of reflexive polytopes  $\longleftrightarrow$ 

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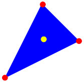
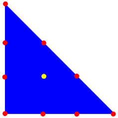
Duality of reflexive polytopes  $\longleftrightarrow$ 

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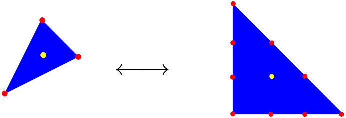

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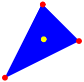
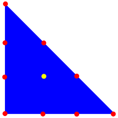
Duality of reflexive polytopes  $\longleftrightarrow$ 

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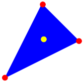
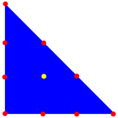
Duality of reflexive polytopes  $\longleftrightarrow$ 

Involution of Gorenstein toric Fano var. polarized by $-K_Y$
Mirror symmetry of anti-canonical hypersurfaces

Associate to Δ monomial degeneration $\mathfrak{X}$ given by

$$I = \left\langle m_0 + t \cdot \sum_{m \in \Delta \cap M} a_m \cdot \varphi_m(m_0) \right\rangle \text{ with } m_0 = \prod_{r \in \Sigma(1)} x_r$$

$\partial\Delta^* \cap N =$ Divisor classes on X induced from Y
$\partial\Delta^* \cap N =$ Polynomial deformations of $X_0^\vee \subset Y$ (not *id*)
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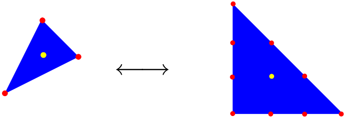
Duality of reflexive polytopes 

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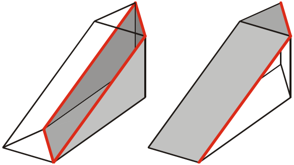

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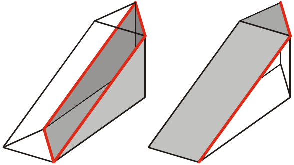

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Integrally affine structures

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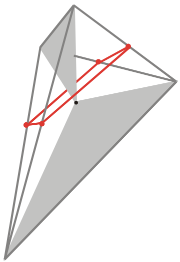

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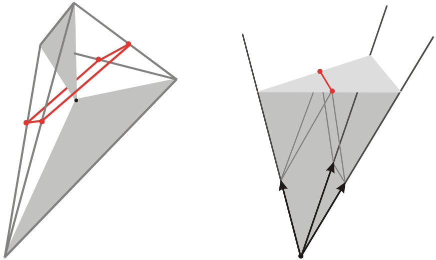

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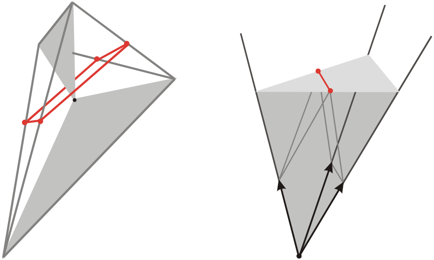

Task: Find fan birational to Y such that (T) is satisfied.

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$$\mathbb{P}^2 \supset X_t = \begin{array}{c} x_0 x_3 + t \cdot [2] = 0 \\ x_1 x_2 + t \cdot [2] = 0 \end{array} \quad \leftrightarrow$$

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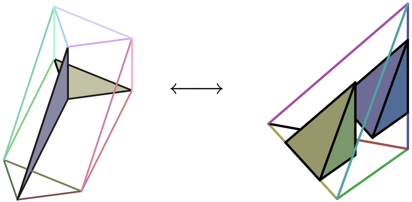
Then tropical mirror $X_t^\vee \subset Y^\vee$ is the degeneration assoc. to dual nef part.

Batyrev-Borisov:
$\nabla = \nabla_1 + \ldots + \nabla_c$
$\nabla_j = \mathrm{convhull}(I_j \cup \{0\})$



Applications: Pfaffian non-complete intersection C(4,7)

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Generic degree 14 Calabi-Yau 3-fold in $\mathbb{P}^6$ given by the 6×6 Pfaffians of

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$A_t : \mathcal{F}^\vee(-1) \to \mathcal{F}$ with $\mathcal{F} = 7\mathcal{O}$

$$A_t = \begin{pmatrix} 0 & 0 & x_0 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & x_3 & 0 & 0 & -x_4 \\ -x_0 & 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & -x_3 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & -x_6 & 0 & 0 & 0 & x_5 \\ x_1 & 0 & 0 & -x_2 & 0 & 0 & 0 \\ 0 & x_4 & 0 & 0 & -x_5 & 0 & 0 \end{pmatrix} + t \cdot \begin{pmatrix} \text{generic} \\ \text{skew} \end{pmatrix}$$

degenerating to Stanley-Reisner ring of $\partial C(4,7)$.

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Recover Rødland's orbifolding mirror.

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Generic degree 13 Pfaffian C-Y 3-fold in $\mathbb{P}^6$

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